MATB44 Exam Notes

Note: The exam is not cumulative.

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Topics covered here are:

1. Variation of Parameters
2. Homogeneous Systems
3. Non-homogeneous Systems
4. 3×3 Linear Systems
5. General Theory of Linear Eqns
6. Series Soln Near An Ordinary Point
7. Series Soln Near A Singular Point
8. Bessel Eqn

Note: The topics listed above are all topics that were not on the midterm. To find topics covered before the midterm, look at my midterm notes.

Note: I don't know what the cutoff for the exam is. The exam may cover some stuff from before the midterm.

Variation of Parameter

- Another way to solve non-homogeneous eqns (N-H eqns).
- Unlike the method of undetermined coefficients, this works for any RHS.
- Recall: A second order n-h diff eqn has the form $P(t)y'' + Q(t)y' + R(t)y = G(t)$ or $y'' + p(t)y' + q(t)y = g(t)$ s.t. $G(t)$ and $g(t)$ are not 0.
- Recall: The general soln of a N-H eqn is the general soln of the corresponding H-eqn + A particular soln of the N-H eqn. We will use variation of parameters to find the particular soln.

E.g. Solve $y'' + y = \frac{1}{\sin t}$

Soln:

First, find the soln to the h-eqn.

$y'' + y = 0$

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$\lambda = 0,\ u = 1$

$y_1 = \cos(t)$

$y_2 = \sin(t)$

Next, we have $y = u_1(t)y_1 + u_2(t)y_2$, where u_1 and u_2 are functions. We want to find u_1 and u_2 through the following 2 eqns:

1 $u_1'y_1 + u_2'y_2 = 0$

2 $u_1'y_1' + u_2'y_2' = \dfrac{1}{\sin t}$

$$u_2' = \frac{-u_1'y_1}{y_2}$$

$$= \frac{-u_1'\cos t}{\sin t}$$

Plug this into 2.

$$u_1'(\cos t)' - \frac{u_1'\cos t(\sin t)'}{\sin t} = \frac{1}{\sin t}$$

$$-u_1'\sin^2 t - u_1'\cos^2 t = 1$$

$$-u_1'(\sin^2 t + \cos^2 t) = 1$$

$$u_1' = -1$$

$$u_1 = \int -1\, dt$$

$$= -t + c_1$$

$$u_2' = \frac{-u_1'\cos t}{\sin t}$$

$$= \frac{\cos t}{\sin t}$$

$$u_2 = \int \frac{\cos t}{\sin t}\, dt$$

Let $u = \sin t$

$\frac{du}{dt} = \cos t \rightarrow dt = \frac{du}{\cos t}$

$$U_2 = \int \frac{1}{u}\,du$$
$$= \ln|u| + C_2$$
$$= \ln|\sin(t)| + C_2$$

$$y = U_1 y_1 + U_2 y_2$$
$$= (-t + C_1)\cos t + (\ln|\sin(t) + C_2)\sin t$$
$$= C_1 \cos t + C_2 \sin t - t\cos t + \ln|\sin t|\sin t$$

E.g. Solve $x^2 y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\sin x$ with $y_1 = x^{-1/2}\sin x$ and $y_2 = x^{-1/2}\cos x$.

Soln:

First, we divide both sides of the eqn by x^2 to make the coefficient of y'' 1.

$$y'' + \frac{y'}{x} + y - \frac{0.25}{x^2}y = 3x^{-1/2}\sin x$$

$$U_1' y_1 + U_2' y_2 = 0$$
$$U_1' y_1' + U_2' y_2' = 3x^{-1/2}\sin x$$

$$U_1' = \frac{-U_2' y_2}{y_1}$$
$$= \frac{-U_2'(x^{-1/2}\cos x)}{x^{-1/2}\sin x}$$
$$= \frac{-U_2'\cos x}{\sin x}$$

$$\left(\frac{-U_2' \cos x}{\sin x}\right)\left(-\frac{x^{-3/2}}{2}\sin x + x^{-1/2}\cos x\right) +$$

$$U_2'\left(-\frac{x^{-3/2}}{2}\cos x - x^{-1/2}\sin x\right) = 3x^{-1/2}\sin x$$

$$U_2'\begin{pmatrix} \frac{x^{-3/2}}{2}\sin x\cos x - x^{-1/2}\cos^2 x \quad - \\ \frac{x^{-3/2}}{2}\sin x\cos x - x^{-1/2}\sin^2 x \end{pmatrix}$$

$$= 3x^{-1/2}\sin^2 x$$

$$U_2' = -3\sin^2 x$$

$$U_2 = \int -3\sin^2 x\, dx$$
$$= \frac{-3x}{2} + \frac{3\sin(2x)}{4} + C_2$$

$$U_1' = \frac{-U_2' \cos x}{\sin x}$$
$$= \frac{3\sin^2 x \cos x}{\sin x}$$
$$= 3\sin x\cos x$$
$$U_1 = \int 3\sin x\cos x\, dx$$
$$= \frac{3\sin^2 x}{2} + C_1$$

$$y = U_1 y_1 + U_2 y_2$$

E.g. Solve $y'' + y = \tan t$

Soln:

$y'' + y = 0$
$r^2 + 1 = 0$
$r^2 = -1$
$r = \pm i$

$\lambda = 0, u = 1$
$y_1 = \cos t, \quad y_2 = \sin t$

$y = U_1 y_1 + U_2 y_2$

$U_1' y_1 + U_2' y_2 = 0$
$U_1' y_1' + U_2' y_2' = \tan t$

$$U_1' = \frac{-U_2' y_2}{y_1}$$
$$= \frac{-U_2' \sin t}{\cos t}$$

$U_1' y_1' + U_2' y_2' = \tan t$
$$\left(\frac{-U_2' \sin t}{\cos t}\right) - \sin t + U_2' \cos t = \tan t$$
$U_2' \sin^2 t + U_2' \cos^2 t = \sin t$
$U_2' (\sin^2 t + \cos^2 t) = \sin t$
$U_2' = \sin t$
$U_2 = -\cos t + C_2$

$$U_1' = \frac{-U_2' \sin t}{\cos t}$$
$$= \frac{-\sin^2 t}{\cos t}$$

$$U_1 = \int \frac{-\sin^2 t}{\cos t} dt$$
$$= \sin t - \log(1 + \sin t) - \log(\cos t) + C_1$$

$$y = U_1 y_1 + U_2 y_2$$

E.g. Solve $x^2 y'' - 3xy' + 4y = x^2 \log x$

Soln:

$x^2 y'' - 3xy' + 4y = 0$ ← Euler Eqn

Recall that with Euler Eqn, $y = x^r$.

$\alpha = -3, \beta = 4$

$$r^2 + (\alpha - 1)r + \beta = 0$$
$$r^2 - 4r + 4 = 0$$
$$(r-2)^2 = 0$$
$$r_1 = r_2 = 2$$
$$y_1 = x^2, \quad y_2 = \ln(x) \cdot x^2$$

Now, I'll rewrite the original eqn so that the coefficient of y'' is 1.

$$y'' - \frac{3y'}{x} + \frac{4}{x^2} y = \log x$$

$y = u_1 y_1 + u_2 y_2$

$u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = \log x$

$u_1' = \frac{-u_2' y_2}{y_1}$

$= \frac{-u_2' \ln(x) \cdot x^2}{x^2}$

$= -u_2' \ln(x)$

$(-u_2' \ln(x))(2x) + u_2'(x + 2x\ln(x)) = \log x$

$-u_2' \ln(x)(2x) + u_2' x + u_2' \ln(x)(2x) = \log x$

$u_2' x = \log x$

$u_2' = \frac{\log x}{x}$

$u_2 = \int \frac{\log x}{x}\, dx$

Let $u = \log x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = du\,x$

$u_2 = \int u\, du$

$= u^2/2 + C_2$

$= \frac{(\log x)^2}{2} + C_2$

$u_1' = -u_2' \ln(x)$

$\quad = -\frac{(\ln(x))^2}{x}$

$u_1 = -\int \frac{(\ln(x))^2}{x}\, dx$

let $u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x}$

$dx = du\, x$

$u_1 = -\int u^2\, du$

$\quad = \frac{-u^3}{3} + c_1$

$\quad = -\frac{(\ln(x))^3}{3} + c_1$

$y = u_1 y_1 + u_2 y_2$

E.g. Solve $y'' - 5y' + 6y = 2e^t$

Soln:

$y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$y_1 = e^{2t}, \ y_2 = e^{3t}$

Let $y = u_1 y_1 + u_2 y_2$

$u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = 2e^t$

$$U_1' = \frac{-U_2' y_2}{y_1}$$
$$= \frac{-U_2' e^{3t}}{e^{2t}}$$
$$= -U_2' e^{t}$$

$$(-U_2' e^{t})(2e^{2t}) + U_2'(3e^{3t}) = 2e^{t}$$
$$U_2'(3e^{3t} - 2e^{3t}) = 2e^{t}$$
$$U_2'(e^{3t}) = 2e^{t}$$
$$U_2' = \frac{2}{e^{2t}}$$
$$U_2 = \int \frac{2}{e^{2t}}\, dt$$
$$= \int 2e^{-2t}\, dt$$

Let $u = -2t$

$$\frac{du}{dt} = -2$$
$$dt = \frac{du}{-2}$$

$$U_2 = -\int e^{u}\, du$$
$$= -e^{u} + C_2$$
$$= -e^{-2t} + C_2$$

$$U_1' = -U_2' e^{t}$$
$$= -2e^{-t}$$

$$U_1 = \int -2e^{-t}\, dt$$
$$= 2e^{-t} + C_1$$

$$y = U_1 y_1 + U_2 y_2$$

$$y = (2e^{-t} + c_1)e^{2t} + (-e^{-2t} + c_2)(e^{3t})$$
$$= 2e^{t} + C_1e^{2t} - e^{t} + C_2e^{3t}$$
$$= C_1e^{2t} + C_2e^{3t} + e^{t}$$

E.g. Solve $2y'' + 18y = 6\tan(3t)$

Soln:

Solve $2y'' + 18y = 0$
$r^2 + 9 = 0$
$r^2 = -9$
$r = \pm 3i$

$\lambda = 0$, $u = 3$
$y_1 = \cos(3t)$, $y_2 = \sin(3t)$

Rewrite the original eqn so that the coefficient of y'' is 1.

$y'' + 9y = 3\tan(3t)$.

Let $y = u_1y_1 + u_2y_2$

$u_1'y_1 + u_2'y_2 = 0$
$u_1'y_1' + u_2'y_2' = 3\tan(3t)$

$$u_1' = \frac{-u_2'y_2}{y_1}$$
$$= \frac{-u_2'\sin(3t)}{\cos(3t)}$$
$$= -u_2'\tan(3t)$$

$-U_2' \tan(3t)(-3\sin(3t)) + U_2'(3\cos(3t)) = 3\tan(3t)$

$3U_2'(\tan(3t)\sin(3t) + \cos(3t)) = 3\tan(3t)$

$U_2'(\sin^2(3t) + \cos^2(3t)) = \sin(3t)$

Note: $\tan(3t) = \frac{\sin(3t)}{\cos(3t)}$

I simply multiplied both sides of the eqn by $\cos(3t)$.

$U_2' = \sin(3t)$

$U_2 = \int \sin(3t)\, dt$

let $u = 3t$

$\frac{du}{dt} = 3$

$dt = \frac{du}{3}$

$$U_2 = \frac{\int \sin(u)}{3}\, du$$

$$= \frac{-\cos(u)}{3} + C_2$$

$$= \frac{-\cos(3t)}{3} + C_2$$

$$U_1' = -U_2' \tan(3t)$$

$$= -\sin(3t)\tan(3t)$$

$$U_1 = \int -\sin(3t)\tan(3t)\, dt$$

$$= \frac{\sin(3t) - \ln(|\tan(3t) + \sec(3t)|)}{3} + C_1$$

$$Y = U_1 y_1 + U_2 Y_2$$

$$= \left(\frac{\sin(3t) - \ln(|\tan(3t) + \sec(3t)|)}{3} + c_1 \right) \cos(3t)$$

$$+ \left(\frac{-\cos(3t)}{3} + c_2 \right) \sin(3t)$$

$$= C_1 \cos(3t) + C_2 \sin(3t) - \frac{\cos(3t)}{3} \ln|\tan(3t) + \sec(3t)|$$

E.g. Solve $y'' - y' - 2y = 2e^{-t}$.

Soln:

Solve $y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$
$(r-2)(r+1) = 0$
$r_1 = 2, \; r_2 = -1$
$y_1 = e^{2t}, \; y_2 = e^{-t}$

$y = U_1 y_1 + U_2 y_2$
$U_1' y_1 + U_2' y_2 = 0$
$U_1' y_1' + U_2' y_2' = 2e^{-t}$

$$U_2' = \frac{-U_1' y_1}{y_2}$$

$$= \frac{-U_1' e^{2t}}{e^{-t}}$$

$$= -U_1' e^{3t}$$

$$u_1' 2e^{2t} + (-u_1' e^{3t})(-e^{-t}) = 2e^{-t}$$
$$u_1' (2e^{2t} + e^{2t}) = 2e^{-t}$$
$$u_1' 3e^{2t} = 2e^{-t}$$
$$u_1' = \frac{2e^{-3t}}{3}$$

$$u_1 = \int \frac{2}{3} e^{-3t} \, dt$$

Lt $u = -3t$

$$\frac{du}{dt} = -3$$
$$dt = \frac{du}{-3}$$

$$u_1 = \frac{-2}{9} \int e^{u} \, du$$
$$= \frac{-2}{9} e^{-3t} + c_1$$

$$u_2' = -u_1' e^{3t}$$
$$= -\frac{2}{3}$$

$$u_2 = -\frac{2}{3}t + c_2$$

$$y = u_1 y_1 + u_2 y_2$$
$$= \left(\frac{-2}{9} e^{-3t} + c_1\right) e^{2t} + \left(-\frac{2}{3}t + c_2\right) e^{-t}$$

$$= c_1 e^{2t} + c_2 e^{-t} - \frac{2}{9} e^{-t} - \frac{2}{3} t e^{-t}$$

Note that $-\frac{2}{9}$ is a constant. Hence, it can be integrated into C_2.

$$Y = C_1 e^{2t} + C_2 e^{-t} - \frac{2}{3} t e^{-t}$$

E.g. Solve $4y'' - 4y' + y = 16e^{t/2}$

Soln:

Solve $4y'' - 4y' + y = 0$

$4r^2 - 4r + 1 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16-16}}{8}$$

$$= \frac{1}{2}$$

$Y_1 = e^{t/2}$, $Y_2 = te^{t/2}$

$y = U_1 y_1 + U_2 y_2$

$U_1' y_1 + U_2' Y_2 = 0$

$U_1' Y_1' + U_2' Y_2' = 4e^{t/2}$

Note: We have to divide the eqn by 4 to make the coefficient of y'' 1.

$$U_1' = \frac{-U_2' Y_2}{Y_1}$$

$$= \frac{-U_2' te^{t/2}}{e^{t/2}}$$

$$= -U_2' t$$

$$\frac{-u_2' t e^{t/2}}{2} + u_2' \left(e^{t/2} + \frac{t}{2} e^{t/2}\right) = 4e^{t/2}$$

$-u_2' t + 2u_2' + u_2' t = 8$

$u_2' = 4$

$u_2 = 4t + C_2$

$u_1' = -u_2' t$

$\quad = -4t$

$u_1 = -2t^2 + C_1$

$y = u_1 y_1 + u_2 y_2$

$\quad = (-2t^2 + C_1)e^{t/2} + (4t + C_2)te^{t/2}$

$\quad = C_1 e^{t/2} - 2t^2 e^{t/2} + 4t^2 e^{t/2} + C_2 t e^{t/2}$

$\quad = C_1 e^{t/2} + C_2 t e^{t/2} + 2t^2 e^{t/2}$

E.g. Solve $y'' + y = \tan(t)$

Soln:

Solve the homogeneous eqn $y'' + y = 0$.

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$\lambda = 0, \ u = 1$

$y_1 = \cos(t), \ y_2 = \sin(t)$

Let $y = U_1 Y_1 + U_2 Y_2$

$U_1' y_1 + U_2' y_2 = 0$

$U_1' y_1' + U_2' y_2' = \tan(t)$

$$U_1' = \frac{-U_2' Y_2}{Y_1}$$

$$= \frac{-U_2' \sin t}{\cos t}$$

$$= -U_2' \tan t$$

$(-U_2' \tan t)(-\sin t) + U_2' \cos t = \tan t$

$U_2' (\sin^2 t + \cos^2 t) = \sin t$

$U_2' = \sin t$

$U_2 = \int \sin t \, dt$

$= -\cos t + C_2$

$U_1' = -\sin t \tan t$

$$U_1 = -\int \frac{\sin^2 t}{\cos t} \, dt$$

$= \ln|\tan(t) + \sec(t)| - \sin(t) + C_1$

$Y = U_1 y_1 + U_2 y_2$

$= (\ln|\tan(t) + \sec(t)| - \sin(t) + C_1)\cos(t) +$
$(-\cos(t) + C_2)\sin(t)$

$= C_1 \cos(t) + C_2 \sin(t) + \cos(t)(\ln|\tan(t) +$
$\sec(t)|)$

E.g. Solve $t^2 y'' - 2y = 3t^2 - 1$

Soln:

First, solve the homogeneous eqn $\underbrace{t^2 y'' - 2y = 0}_{\text{Euler Eqn}}$.

$\alpha = 0, \beta = -2$

$r^2 + (\alpha - 1)r + \beta = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2, r_2 = -1$

$y_1 = t^{r_1} \qquad y_2 = t^{r_2}$

$ = t^2 \qquad = t^{-1}$

Let $Y = U_1 Y_1 + U_2 Y_2$

$U_1' y_1 + U_2' Y_2 = 0$

$U_1' y_1' + U_2' Y_2' = 3 - \frac{1}{t^2}$ ← Note: We get $3 - \frac{1}{t^2}$ when we divide both sides of the eqn by t^2.

$$U_2' = \frac{-U_1' Y_1}{Y_2}$$

$$= \frac{-U_1' t^2}{t^{-1}}$$

$$= -U_1' t^3$$

$$U_1' 2t - U_1' t^3 (-t^{-2}) = 3 - t^{-2}$$
$$U_1' 2t + U_1' t = 3 - t^{-2}$$
$$U_1' 3t = 3 - t^{-2}$$
$$U_1' = \frac{1}{t} - \frac{1}{3t^3}$$

$$U_1 = \int \frac{1}{t} - \frac{1}{3t^3} \, dt$$

$$= \ln|t| + \frac{1}{6t^2} + c_1$$

$$U_2' = -U_1' t^3$$
$$= -t^2 + \frac{1}{3}$$

$$U_2 = \int -t^2 + \frac{1}{3} \, dt$$

$$= \frac{-t^3}{3} + \frac{t}{3} + c_2$$

$$Y = U_1 Y_1 + U_2 Y_2$$
$$= \left(\ln|t| + \frac{1}{6t^2} + c_1 \right) t^2 + \left(-\frac{t^3}{3} + \frac{t}{3} + c_2 \right) t^{-1}$$

$$= c_1 t^2 + c_2 t^{-1} + t^2 \ln|t| + \frac{1}{6} - \frac{t^2}{3} + \frac{1}{3}$$

$$= c_1 t^2 + c_2 t^{-1} + t^2 \ln|t| + \frac{1}{2}$$

E.g. Solve $y'' - 2y' + 2y = 2e^{t}\cos t - e^{t}\sin t$

Soln:

First, solve $y'' - 2y' + 2y = 0$.

$r^2 - 2r + 2 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

$\lambda = 1$, $u = 1$

$Y_1 = e^{t}\cos(t)$, $Y_2 = e^{t}\sin(t)$

Let $Y = U_1Y_1 + U_2Y_2$

$U_1'Y_1 + U_2'Y_2 = 0$

$U_1'Y_1' + U_2'Y_2' = 2e^{t}\cos(t) - e^{t}\sin(t)$

$$U_1' = \frac{-U_2'Y_2}{Y_1}$$

$$= \frac{-U_2' e^{t}\sin(t)}{e^{t}\cos(t)}$$

$$= -U_2'\tan(t)$$

$(-U_2'\tan(t))(e^{t}\cos(t) - e^{t}\sin(t)) +$

$U_2'(e^{t}\sin(t) + e^{t}\cos(t)) = 2e^{t}\cos(t) - e^{t}\sin(t)$

$$\left(-\frac{U_2' \sin(t)}{\cos(t)}\right)(\cos(t)-\sin(t)) +$$

$$U_2'(\sin(t)+\cos(t)) = 2\cos(t)-\sin(t)$$

$$(-U_2'\sin(t))(\cos(t)-\sin(t)) + U_2'(\sin(t)\cos(t)+\cos^2(t))$$
$$= 2\cos^2(t) - \sin(t)\cos(t)$$

$$U_2'(\sin(t)\cos(t)+\cos^2(t)-\sin(t)\cos(t)+\sin^2(t))$$

$$= 2\cos^2(t) - \sin(t)\cos(t)$$

$$U_2' = 2\cos^2(t) - \sin(t)\cos(t)$$

$$U_2 = \int 2\cos^2(t) - \sin(t)\cos(t) \; dt$$
$$= -\frac{\sin^2(t)}{2} + \sin(t)\cos(t) + t + C_2$$

$$U_1' = -U_2' \tan(t)$$
$$= -(2\cos^2(t) - \sin(t)\cos(t))\frac{\sin(t)}{\cos(t)}$$
$$= -2\sin(t)\cos(t) + \sin^2(t)$$

$$U_1 = \int -2\sin(t)\cos(t) + \sin^2(t) \; dt$$
$$= -\frac{1}{4}\sin(2t) + \cos^2(t) + \frac{t}{2} + C_1$$

$$Y = U_1Y_1 + U_2Y_2$$

$$y = \left(-\frac{1}{4}\sin(2t) + \cos^2(t) + \frac{t}{2} + C_1\right)e^t\cos(t) +$$
$$\left(-\frac{\sin^2(t)}{2} + \sin(t)\cos(t) + t + C_2\right)e^t\sin(t)$$

E.g. Solve $y'' + y = \dfrac{1}{\cos^3(t)}$

Soln:

Solve the homogeneous eqn $y'' + y = 0$.

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$\lambda = 0,\ u = 1$

$Y_1 = \cos(t),\ Y_2 = \sin(t)$

Let $Y = U_1Y_1 + U_2Y_2$

$U_1'Y_1 + U_2'Y_2 = 0$

$U_1'Y_1' + U_2'Y_2' = \dfrac{1}{\cos^3(t)}$

$$U_1' = \frac{-U_2'Y_2}{Y_1}$$
$$= \frac{-U_2'\sin(t)}{\cos(t)}$$
$$= -U_2'\tan(t)$$

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$$(-U_2' \tan(t))(-\sin(t)) + U_2'(\cos(t)) = \frac{1}{\cos^3(t)}$$

$$(U_2')\sin^2(t) + U_2'\cos^2 t = \frac{1}{\cos^2 t}$$

$$U_2'(\sin^2(t) + \cos^2(t)) = \frac{1}{\cos^2 t}$$

$$U_2' = \frac{1}{\cos^2 t} \longrightarrow U_2 = \int \frac{1}{\cos^2 t}\, dt$$

$$U_2 = \tan(t) + C_2$$

$$U_1' = -U_2' \tan(t)$$
$$= \frac{-\sin(t)}{\cos^3(t)}$$

$$U_1 = -\int \frac{\sin(t)}{\cos^3(t)}\, dt$$

$$= \frac{-1}{2\cos^2 t} + C_1$$

$$Y = U_1 Y_1 + U_2 Y_2$$
$$= \left(\frac{-1}{2\cos^2 t} + C_1\right)\cos(t) + \left(\tan(t) + C_2\right)\sin(t)$$

E.g. Solve $y'' + 2y' + y = \frac{e^{-x}}{x}$

Soln:

Solve $y'' + 2y' + y = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r_1 = r_2 = -1$

$Y_1 = e^{-x}, \ y_2 = xe^{-x}$

Let $Y = U_1Y_1 + U_2Y_2$

$U_1'Y_1 + U_2'Y_2 = 0$

$U_1'Y_1' + U_2'Y_2' = \frac{e^{-x}}{x}$

$$U_1' = \frac{-U_2'Y_2}{Y_1}$$

$$= \frac{-U_2'xe^{-x}}{e^{-x}}$$

$$= -U_2'x$$

$(-U_2'x)(-e^{-x}) + U_2'(e^{-x} - xe^{-x}) = \frac{e^{-x}}{x}$

$U_2'x + U_2'(1-x) = \frac{1}{x}$

$U_2'x + U_2' - U_2'x = x^{-1}$

$U_2' = x^{-1}$

$U_2 = \int x^{-1}\,dx$

$= \ln|x| + C_2$

$u_1' = -u_2' x$
$= -1$

$u_1 = \int -1 \, dx$
$= -x + C_1$

$y = u_1 y_1 + u_2 y_2$
$= (-x + C_1) e^{-x} + (\ln|x| + C_2) x e^{-x}$

E.g. Solve $2x^2 y'' + 3xy' - y = \frac{1}{x}$

Soln:

$x^2 y'' + \frac{3xy'}{2} - \frac{y}{2} = 0$ ← Euler Eqn

$\alpha = \frac{3}{2}, \ \beta = -\frac{1}{2}$
$r^2 + (\alpha - 1) r + \beta = 0$
$r^2 + \frac{1}{2} r - \frac{1}{2} = 0$
$2r^2 + r - 1 = 0$
$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$= \frac{-1 \pm \sqrt{1 + 8}}{4}$
$= \frac{-1 \pm 3}{4}$
$= \frac{1}{2}$ or -1

$y_1 = x^{r_1}$ $\quad$ $y_2 = x^{r_2}$
$= x^{1/2}$ $\quad$ $= x^{-1}$

Now, let's divide both sides of the eqn by $2x^2$ to make the coefficient of y'' 1.

$$y'' + \frac{3y'}{2x} - \frac{y}{2x^2} = \frac{1}{2x^3}$$

Let $Y = U_1 Y_1 + U_2 Y_2$

$U_1' Y_1 + U_2' Y_2 = 0$

$U_1' Y_1' + U_2' Y_2' = \frac{1}{2x^3}$

$$U_2' = \frac{-U_1' y_1}{y_2}$$

$$= \frac{-U_1' x^{1/2}}{x^{-1}}$$

$$= -U_1' x^{3/2}$$

$$U_1' \frac{x^{-1/2}}{2} + U_1' x^{3/2} x^{-2} = \frac{1}{2x^3}$$

$$U_1' x^{-1/2} + 2U_1' x^{-1/2} = x^{-3}$$

$$U_1' = \frac{x^{-5/2}}{3}$$

$$U_1 = \int \frac{x^{-5/2}}{3} \, dx$$

$$= -\frac{2}{9} x^{-3/2} + C_1$$

$$U_2' = -U_1' x^{3/2}$$
$$= \frac{-x^{-1}}{3}$$

$$U_2 = \int \frac{-x^{-1}}{3}\, dx$$
$$= \frac{-\ln|x|}{3} + C_2$$

$$Y = U_1Y_1 + U_2Y_2$$
$$= \left(-\frac{2}{9}x^{-3/2} + C_1\right)x^{1/2} + \left(\frac{-\ln|x|}{3} + C_2\right)x^{-1}$$

System of Linear Eqns With Constant Coefficients:

- Has the form $\overline{x'} = A\bar{x} + \bar{g}$
- Note: We say the system is homogeneous if $\bar{g} = \bar{0}$ and non-homogeneous if $\bar{g} \neq \bar{0}$.

Homogeneous Systems:

- Has the form $\overline{x'} = A\bar{x}$.
- Let $\bar{x} = e^{rt}\bar{z}$ where $\bar{z}$ is a vector.

$\bar{x}' = re^{rt}\bar{z}$

$A\bar{x}$ $e^{rt} A\bar{z}$

$\bar{x}' = A\bar{x}$

$re^{rt}\bar{z} = e^{rt}A\bar{z}$

$r\bar{z} = A\bar{z}$

$(A - rI)\bar{z} = 0$

- The system has a non-trivial soln iff $\det(A - rI) = 0$.

- A homogeneous system has unique, non-trivial solns iff $\det(A - rI) = 0$.

- As you will see, we will be using the quadratic eqn to find the eigenvalues. Hence, there are 3 cases:

a) 2 real, distinct eigenvalues
b) Repeated eigenvalues
c) Complex eigenvalues

- In all cases, we'll have 2 eigenvalues and 2 eigenvectors. Each eigenvalue corresponds to an eigenvector.

Case 1: 2 real, distinct eigenvalues

E.g. Solve $\bar{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = 0$$

Note: The characteristic eqn is $\det(A-\lambda I)=0$.

$(-2-r)^2 - 1 = 0$

$r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$r_1 = -3,\ r_2 = -1$

$(A - rI)\bar{z} = \bar{0}$ ← Eigenvector Eqn

$r = -3$

$$\begin{bmatrix} -2+3 & 1 \\ 1 & -2+3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 + z_2 = 0$

$z_1 + z_2 = 0$ ← Redundant

Recall: If matrix A is non-singular, i.e. $\det(A)=0$, then the rows of matrix A are linearly dependent. Hence, we will always get a redundant eqn.

$z_1 = -z_2$

Let $z_1 = 1$. Then, $z_2 = -1$.

$$\bar{z}^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

When $r=-1$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-z_1 + z_2 = 0$

$z_1 - z_2 = 0.$ ← Redundant

$z_1 = z_2$

Let $z_1 = 1$. Then, $z_2 = 1$.

$$\overline{z^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note: $\overline{z^1}$ and $\overline{z^2}$ are always linearly dependent.

$$\bar{x} = C_1 e^{r_1 t} \overline{z^1} + C_2 e^{r_2 t} \overline{z^2}$$
$$= C_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0$$

$(1-r)^2 - 4 = 0$

$r^2 - 2r + 1 - 4 = 0$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r_1 = 3, \; r_2 = -1$

$(A - rI)\bar{z} = \bar{0}$

when $r = 3$

$$\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2z_1 + z_2 = 0$

$4z_1 - 2z_2 = 0$ ← Redundant

$2z_1 = z_2$

Let $z_1 = 1$. $z_2 = 2$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

when $r = -1$

$$\begin{bmatrix} 1+1 & 1 \\ 4 & 1+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2z_1 + z_2 = 0$

$4z_1 + 2z_2 = 0$ ← Redundant

$2z_1 = -z_2$

Let $z_1 = 1$. $z_2 = -2$

$$\overline{z^2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{aligned} \bar{x} &= c_1 e^{r_1 t} \overline{z^1} + c_2 e^{r_2 t} \overline{z^2} \\ &= c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{aligned}$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} = 0$$

$(3-r)(-2-r)+4=0$

$-6-3r+2r+r^2+4=0$

$r^2-r-2=0$

$(r-2)(r+1)=0$

$r_1=2,\ r_2=-1$

$(A-rI)\bar{z}=\bar{0}$

when $r=2$

$$\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - 2z_2 = 0$

$2z_1 - 4z_2 = 0$ ← Redundant

$z_1 = 2z_2$

let $z_1 = 2$.

$z_2 = 1$.

$\bar{z}' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

When $r = -1$

$(A - rI)\bar{z} = \bar{0}$

$$\begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 - 2z_2 = 0$
$2z_1 - z_2 = 0$

$4z_1 = 2z_2$
$2z_1 = z_2$
Let $z_1 = 1$. $z_2 = 2$

$$\overline{z^2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\bar{x} = C_1 e^{r_1 t} \overline{z^1} + C_2 e^{r_2 t} \overline{z^2}$$
$$= C_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Case 2: Repeated Eigenvalues

E.g. Solve $\overline{x'} = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \overline{x}$

Soln:

$$\begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = 0$$

$(1-r)(-7-r)+16=0$

$-7-r+7r+r^2+16=0$

$r^2+6r+9=0$

$(r+3)^2=0$

$r_1=r_2=-3$

$(A-rI)\overline{z}=\overline{0}$

$$\begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 - 4z_2 = 0$

$4z_1 - 4z_2 = 0$ ← Redundant

$z_1 = z_2$

let $z_1 = 1$. $z_2 = 1$.

$$\overline{z^1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find the second eigenvector, we need a generalized eigenvector.

$X_1 = e^{r_1 t}\overline{z'}$
$= e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$X_2 = te^{r_1 t}\overline{z'} + e^{r_1 t}\overline{p}$, where p is an unknown vector.

$(A - rI)\overline{p} = \overline{z}$ is called the generalized eigenvector eqn.

$r = -3$

$$\begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$4P_1 - 4P_2 = 1$
$4P_1 - 4P_2 = 1$ ← Redundant

$P_1 - P_2 = 1/4$

Note: We can let P_1 or P_2 be 0 because we have a non-homogeneous system. If we have a homogeneous system, we can't let $X_i = 0$.

We can choose a few different values for P_1 or P_2. If we get 2 different $\overline{P}$'s, their difference will be proportional to $\overline{z}$.

Let $P_1 = 0$. $P_2 = -1/4$
$\overline{p} = \begin{bmatrix} 0 \\ -1/4 \end{bmatrix}$

$$\bar{x} = C_1 e^{r_1 t}\bar{z'} + C_2(te^{r_1 t}\bar{z'} + e^{r_1 t}\bar{p})$$
$$= C_1 e^{-3t}\begin{bmatrix}1\\1\end{bmatrix} + C_2\left(te^{-3t}\begin{bmatrix}1\\1\end{bmatrix} + e^{-3t}\begin{bmatrix}0\\-1/4\end{bmatrix}\right)$$

Note: $\bar{z'}$ and $\bar{p}$ are linearly independent.

Case 3: Complex Eigenvalues

- When we have complex eigenvalues, we have complex eigenvectors.

E.g. Solve $\bar{x'} = \begin{bmatrix}1 & 2\\-5 & -1\end{bmatrix}\bar{x}$

Soln:

$$\begin{vmatrix}1-r & 2\\-5 & -1-r\end{vmatrix} = 0$$

$(1-r)(-1-r)+10=0$

$-1-r+r+r^2+10=0$

$r^2+9=0$

$r^2=-9$

$r=\pm 3i$

$(A-rI)\bar{z}=0$

when $r=3i$

$$\begin{bmatrix}1-3i & 2\\-5 & -1-3i\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$(1-3i)z_1 + 2z_2 = 0$

$-5z_1 + (-1-3i)z_2 = 0$ ← Redundant

$(1-3i)z_1 = -2z_2$

$\frac{(1-3i)z_1}{-2} = z_2$

Let $z_1 = 2$, $z_2 = -1+3i$

$$\overline{z'} = \begin{bmatrix} 2 \\ -1+3i \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\bar{x} = e^{r_1 t}\,\overline{z'}$$
$$= e^{(3t)i}\,\overline{z'}$$
$$= (\cos(3t) + i\sin(3t))\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

$$= \cos(3t)\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) - \sin(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix} +$$

$$i\left(\cos(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix} + \sin(3t)\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$$

General soln:

$$\bar{x} = C_1\left(\cos(3t)\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \sin(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) +$$

$$C_2\left(\cos(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix} + \sin(3t)\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$$

More Examples:

E.g. Solve $\bar{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} = 0$$

$(3-r)(-2-r)+4=0$

$-6-3r+2r+r^2+4=0$

$r^2-r-2=0$

$(r-2)(r+1)=0$

$r_1=2, \quad r_2=-1$

$(A-rI)\bar{z}=\bar{0}$

when $r=2$

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - 2z_2 = 0$

$z_1 = 2z_2$

Let $z_1=2. \ z_2=1$

$$\overline{z'} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

When $r=-1$

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 - 2z_2 = 0$
$2z_1 - z_2 = 0$

$2z_1 = z_2$
Let $z_1 = 1.\ z_2 = 2.$

$$\overline{z^2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \overline{x} &= C_1 e^{r_1 t}\, \overline{z^1} + C_2 e^{r_2 t}\, \overline{z^2} \\ &= C_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

E.g. Solve $\overline{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \overline{x}$

Soln:

$$\begin{vmatrix} 1-r & -2 \\ 3 & -4-r \end{vmatrix} = 0$$

$(1-r)(-4-r)+6=0$
$-4-r+4r+r^2+6=0$
$r^2+3r+2=0$
$(r+2)(r+1)=0$
$r_1=-2,\ r_2=-1$

$(A - rI)\bar{z} = \bar{0}$

when $r = -2$

$$\begin{bmatrix} 1+2 & -2 \\ 3 & -4+2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - 2z_2 = 0$

$3z_1 - 2z_2 = 0$

$\frac{3z_1}{2} = z_2$

let $z_1 = 2$. $z_2 = 3$.

$$\bar{z^1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

when $r = -1$

$$\begin{bmatrix} 1+1 & -2 \\ 3 & -4+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2z_1 - 2z_2 = 0$

$3z_1 - 3z_2 = 0$

$z_1 = z_2$

let $z_1 = 1$. $z_2 = 1$.

$$\bar{z^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \bar{x} = C_1 e^{r_1 t} \bar{z^1} + C_2 e^{r_2 t} \bar{z^2}$$

$$= C_1 e^{-2t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

E.g. Solve $\overline{x'} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \overline{x}$

Soln:

$$\begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = 0$$

$(2-r)(-2-r)+3=0$

$-4-2r+2r+r^2+3=0$

$r^2-1=0$

$r^2=1$

$r=\pm 1 \rightarrow r_1=1, \quad r_2=-1$

$(A-rI)\overline{z}=\overline{0}$

When $r=1$

$$\begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - z_2 = 0$

$3z_1 - 3z_2 = 0$

$z_1 = z_2$

let $z_1=1.$ $z_2=1.$

$\overline{z^1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

When $r=-1$

$$\begin{bmatrix} 2+1 & -1 \\ 3 & -2+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - z_2 = 0$
$3z_1 - z_2 = 0$

$3z_1 = z_2$
Let $z_1 = 1.$ $z_2 = 3$

$$\bar{z}^1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$x = C_1 e^{r_1 t}\,\bar{z}^1 + C_2 e^{r_2 t}\,\bar{z}^2$$
$$= C_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \bar{x}$, $\bar{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Soln:

$$\begin{vmatrix} 5-r & -1 \\ 3 & 1-r \end{vmatrix} = 0$$

$(5-r)(1-r) + 3 = 0$
$5 - 5r - r + r^2 + 3 = 0$
$r^2 - 6r + 8 = 0$
$(r-2)(r-4) = 0$
$r_1 = 2$, $r_2 = 4$

$(A-rI)\bar{z}=\bar{0}$

$r=2$

$$\begin{bmatrix} 5-2 & -1 \\ 3 & 1-2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - z_2 = 0$

$3z_1 - z_2 = 0$

$3z_1 = z_2$

let $z_1 = 1.$ $z_2 = 3.$

$$\overline{z^1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$r=4$

$$\begin{bmatrix} 5-4 & -1 \\ 3 & 1-4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - z_2 = 0$

$3z_1 - 3z_2 = 0$

$z_1 = z_2 \rightarrow$ let $z_1 = 1.$ $z_2 = 1$

$$\overline{z^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \bar{x} &= C_1 e^{r_1 t}\,\overline{z^1} + C_2 e^{r_2 t}\,\overline{z^2} \\ &= C_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$\bar{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$2 = C_1 + C_2$
$-1 = 3C_1 + C_2$

$-2C_1 = 3$
$C_1 = \frac{-3}{2}$

$C_2 = 2 - C_1$
$= 2 + \frac{3}{2}$
$= \frac{7}{2}$

$$\bar{x} = \frac{-3}{2} e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{7}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

E.g. Solve $\bar{x}' = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix} \bar{x}$, $\bar{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Soln:

$\begin{vmatrix} -2-r & 1 \\ -5 & 4-r \end{vmatrix} = 0$

$(-2-r)(4-r) + 5 = 0$
$-8 + 2r - 4r + r^2 + 5 = 0$
$r^2 - 2r - 3 = 0$
$(r-3)(r+1) = 0$
$r_1 = 3, \; r_2 = -1$

$(A-rI)\bar{z}=\bar{0}$
when $r=3$

$$\begin{bmatrix} -2-3 & 1 \\ -5 & 4-3 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-5z_1 + z_2=0$
$-5z_1 + z_2=0$

$5z_1=z_2$
let $z_1=1$. $z_2=5$.

$$\overline{z^1}=\begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

when $r=-1$

$$\begin{bmatrix} -2+1 & 1 \\ -5 & 4+1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-z_1 + z_2=0$
$-5z_1 + 5z_2=0$
$-z_1=-z_2$
$z_1=z_2$
let $z_1=1$. $z_2=1$.

$$\overline{z^2}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{x} = C_1 e^{r_1 t}\,\overline{z^1} + C_2 e^{r_2 t}\,\overline{z^2}$$
$$= C_1 e^{3t}\begin{bmatrix}1\\5\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\bar{x}(0) = C_1\begin{bmatrix}1\\5\end{bmatrix} + C_2\begin{bmatrix}1\\1\end{bmatrix}$$

$$\left.\begin{aligned}C_1 + C_2 &= 1\\ 5C_1 + C_2 &= 3\end{aligned}\right\}\ \bar{x}(0) = \begin{bmatrix}1\\3\end{bmatrix}$$

$C_1 = \frac{1}{2}$

$C_2 = \frac{1}{2}$

$$\bar{x} = \tfrac{1}{2}e^{3t}\begin{bmatrix}1\\5\end{bmatrix} + \tfrac{1}{2}e^{-t}\begin{bmatrix}1\\1\end{bmatrix}$$

E.g. Solve $\bar{x}' = \begin{bmatrix}-1 & -4\\ 1 & -1\end{bmatrix}\bar{x}$

Soln:

$$\begin{vmatrix}-1-r & -4\\ 1 & -1-r\end{vmatrix} = 0$$

$(-1-r)^2 + 4 = 0$

$r^2 + 2r + 1 + 4 = 0$

$r^2 + 2r + 5 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

$(A - r I)\bar{z} = \bar{0}$

When $r = -1+2i$

$$\begin{bmatrix} -1-(-1+2i) & -4 \\ 1 & -1-(-1+2i) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2iz_1 - 4z_2 = 0$$
$$z_1 - 2iz_2 = 0$$

$$-2iz_1 = 4z_2$$
$$-\frac{iz_1}{2} = z_2$$

Let $z_1 = 2.\ z_2 = -i$

$$\overline{z^1} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$e^{r_1 t}\overline{z^1} = e^{(-1+2i)t}\overline{z^1}$$
$$= e^{-t} \cdot e^{(2t)i} \cdot \overline{z^1}$$
$$= e^{-t}(\cos(2t) + i\sin(2t))\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$
$$= e^{-t}\left(\cos(2t)\begin{bmatrix} 2 \\ 0 \end{bmatrix} - \sin(2t)\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) +$$
$$ie^{-t}\left(\cos(2t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(2t)\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$$

$$\bar{x} = C_1 e^{-t}\left(\cos(2t)\begin{bmatrix}2\\0\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\-1\end{bmatrix}\right) +$$

$$C_2 e^{-t}\left(\cos(2t)\begin{bmatrix}0\\-1\end{bmatrix} + \sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right)$$

E.g. Solve $\bar{x}' = \begin{bmatrix}2 & -5\\1 & -2\end{bmatrix}\bar{x}$

Soln:

$\begin{vmatrix}2-r & -5\\1 & -2-r\end{vmatrix} = 0$

$(2-r)(-2-r)+5=0$

$-4-2r+2r+r^2+5=0$

$r^2+1=0$

$r^2=-1$

$r=\pm i$

$(A-rI)\bar{z}=0$

when $r=i$

$\begin{bmatrix}2-i & -5\\1 & -2-i\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$(2-i)z_1 - 5z_2 = 0$

$z_1 + (-2-i)z_2 = 0$

$\frac{(2-i)z_1}{5} = z_2$

let $z_1 = 5.\ z_2 = 2-i$

$$\overline{z'} = \begin{bmatrix} 5 \\ 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$e^{r_1 t}\overline{z'}$$

$$= e^{(\ t)i}\,\overline{z'}$$

$$= (\cos(\ t) + i\sin(\ t))\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$$

$$= \cos(\ t)\begin{bmatrix} 5 \\ 2 \end{bmatrix} - \sin(\ t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} +$$

$$i\left(\cos(\ t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(\ t)\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)$$

$$\bar{x} = C_1\left(\cos(\ t)\begin{bmatrix} 5 \\ 2 \end{bmatrix} - \sin(\ t)\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) +$$

$$C_2\left(\cos(\ t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(\ t)\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & -1 \\ 5 & -3-r \end{vmatrix} = 0$$

$(1-r)(-3-r) + 5 = 0$

$-3 - r + 3r + r^2 + 5 = 0$

$r^2 + 2r + 2 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$= -1 \pm i$$

$(A - rI)\bar{z} = \bar{0}$ → Take $r = -1+i$

$$\begin{bmatrix} 1-(-1+i) & -1 \\ 5 & -3-(-1+i) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(2-i)z_1 - z_2 = 0$

$5z_1 + (-3-(-1+i))z_2 = 0$

$(2-i)z_1 = z_2$

Let $z_1 = 1$. $z_2 = 2-i$

$$\bar{z}^1 = \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$e^{r_1 t} \overline{z^1}$

$= e^{(-1+i)t} \overline{z^1}$

$= e^{-t} \cdot e^{it} \cdot \overline{z^1}$

$= e^{-t} (\cos(t) + i\sin(t)) \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$

$= e^{-t} \left(\cos(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) +$

$i e^{-t} \left(\cos(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

$\overline{x} = C_1 e^{-t} \left(\cos(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) +$

$C_2 e^{-t} \left(\cos(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

E.g. Solve $\overline{x'} = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \overline{x}$

Soln:

$\begin{vmatrix} 1-r & 2 \\ -5 & -1-r \end{vmatrix} = 0$

$(1-r)(-1-r) + 10 = 0$

$-1 - r + r + r^2 + 10 = 0$

$r^2 + 9 = 0$

$r^2 = -9$

$r = \pm 3i$

$(A - rI)\bar{z} = \bar{0}$

when $r = 3i$

$$\begin{bmatrix} 1-3i & 2 \\ -5 & -1-3i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(1-3i)z_1 + 2z_2 = 0$

$-5z_1 + (-1-3i)z_2 = 0$

$(1-3i)z_1 = -2z_2$

$\frac{(1-3i)z_1}{-2} = z_2$

let $z_1 = 2$. $z_2 = -(1-3i)$

$= -1+3i$

$$\overline{z_1} = \begin{bmatrix} 2 \\ -1+3i \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$e^{r_1 t}\overline{z_1}$

$= e^{(3i)t}\overline{z_1}$

$$=(\cos(3t) + i\sin(3t))\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

$$= \cos(3t)\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \sin(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix} +$$

$$i\left(\cos(3t)\begin{bmatrix} 0 \\ 3 \end{bmatrix} + \sin(3t)\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$$

$$\bar{x} = C_1\left(\cos(3t)\begin{bmatrix}2\\-1\end{bmatrix} - \sin(3t)\begin{bmatrix}0\\3\end{bmatrix}\right) +$$

$$C_2\left(\cos(3t)\begin{bmatrix}0\\3\end{bmatrix} + \sin(3t)\begin{bmatrix}2\\-1\end{bmatrix}\right)$$

E.g. Solve $\bar{x}' = \begin{bmatrix}3 & -4\\1 & -1\end{bmatrix}\bar{x}$

Soln:

$\begin{vmatrix}3-r & -4\\1 & -1-r\end{vmatrix} = 0$

$(3-r)(-1-r)+4=0$

$-3-3r+r+r^2+4=0$

$r^2-2+1=0$

$(r-1)^2=0$

$r_1=1$

$(A-rI)\bar{z}=\bar{0}$

When $r=1$

$\begin{bmatrix}3-1 & -4\\1 & -1-1\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$2z_1 - 4z_2 = 0$

$z_1 - 2z_2 = 0$

$z_1 = 2z_2$

$\frac{z_1}{2} = z_2$

Let $z_1=2$. $z_2=1$. $\bar{z}^1 = \begin{bmatrix}2\\1\end{bmatrix}$

$$\bar{x} = C_1 e^{r_1 t} \bar{z'} + C_2 (t e^{r_1 t} \bar{z'} + e^{r_1 t} \bar{p})$$

$(A - rI)\bar{p} = \bar{z}$

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$2P_1 - 4P_2 = 2$

$P_1 - 2P_2 = 1$

Let $P_2 = 0$. $P_1 = 1$

$$\bar{P} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{x} = C_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left(t e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 4-r & -2 \\ 8 & -4-r \end{vmatrix} = 0$$

$(4-r)(-4-r) + 16 = 0$

$-16 - 4r + 4r + r^2 + 16 = 0$

$r^2 = 0$

$r = 0$

$(A-rI)\bar{z}=\bar{0} \rightarrow$ when $r=0$

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 - 2z_2 = 0$

$8z_1 - 4z_2 = 0$

$4z_1 = 2z_2$

$2z_1 = z_2$

let $z_1 = 1$. $z_2 = 2$.

$$\overline{z'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$(A-rI)\bar{p} = \bar{z}$

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$4p_1 - 2p_2 = 1$

$8p_1 - 4p_2 = 2$

let $p_2 = 0$.

$p_1 = 1/4$

$$\bar{p} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$$

$$\bar{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \left(t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \right)$$

Non-Homogeneous Linear Systems

- Has the form $\overline{x}' = A\overline{x} + \overline{g}$, where $\overline{g} \neq \overline{0}$.
- The general soln of a N-H linear system equals the general soln of a H linear system + a particular soln of the N-H linear system.
- To find the particular soln, we will use variation of parameters.

E.g. Solve $\overline{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \overline{x} + \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$

Soln:

First, we find the general soln of the homogeneous linear system.

$\begin{vmatrix} -2-r & 1 \\ 1 & -2-r \end{vmatrix} = 0$ ← Characteristic Eqn

$(-2-r)^2 - 1 = 0$

$r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$r_1 = -3, \quad r_2 = -1$

$(A - rI)\overline{z} = \overline{0}$ ← Eigenvector Eqn

when $r = -3$

$\begin{bmatrix} -2+3 & 1 \\ 1 & -2+3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$z_1 + z_2 = 0$

$z_1 + z_2 = 0$ ← Redundant

$z_1 = -z_2$

let $z_1 = 1$. $z_2 = -1$. $\rightarrow \overline{z}' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

When $r=-1$

$$\begin{bmatrix} -2+1 & 1 \\ 1 & -2+1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-z_1 + z_2 = 0$$
$$z_1 - z_2 = 0$$

$z_1 = z_2$

Let $z_1 = 1$. $z_2 = 1$.

$$\overline{z^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} \bar{x} &= C_1 e^{r_1 t}\,\overline{z^1} + C_2 e^{r_2 t}\,\overline{z^2} \\ &= C_1 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}\right\} \text{General Soln of Homogeneous Linear System}$$

Now, we will use variation of parameter to find a particular soln to the N-H linear system.

$U_1 e^{r_1 t}\,\overline{z^1} + U_2 e^{r_2 t}\,\overline{z^2}$ ← Want to find U_1 and U_2.

To find U_1 and U_2, we do

$$U_1' e^{r_1 t}\,\overline{z^1} + U_2' e^{r_2 t}\,\overline{z^2} = \overline{g}$$

In our case, we have

$$U_1' e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + U_2' e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ 3t \end{bmatrix}$$

$u_1' e^{-3t} + u_2' e^{-t} = 2e^{-t}$ (1)
$-u_1' e^{-3t} + u_2' e^{-t} = 3t$ (2)

Do (1) + (2)
$2u_2' e^{-t} = 2e^{-t} + 3t$
$u_2' = 1 + \frac{3}{2} te^{t}$

$u_2 = \int 1 + \frac{3}{2} te^{t} \, dt$

$= t + \frac{3}{2}\left(te^{t} - e^{t}\right) + c_2$

$u_1' e^{-3t} + \left(1 + \frac{3}{2} te^{t}\right)e^{-t} = 2e^{-t}$

$u_1' e^{-3t} + e^{-t} + \frac{3}{2} t = 2e^{-t}$

$u_1' e^{-3t} = e^{-t} - \frac{3}{2} t$

$u_1' = e^{2t} - \frac{3}{2} te^{3t}$

$u_1 = \int e^{2t} - \frac{3}{2} te^{3t} \, dt$

$= \frac{e^{2t}}{2} - \frac{(3t-1)e^{3t}}{6} + c_1$

$$U_1 e^{r_1 t} \overline{z_1} + U_2 e^{r_2 t} \overline{z_2}$$

$$= \left(\frac{e^{2t}}{2} - \frac{(3t-1)e^{3t}}{6} + c_1 \right) e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} +$$

$$\left(t + \frac{3}{2} te^{t} - \frac{3}{2} e^{t} + c_2 \right) e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

E.g. Solve $\overline{x'} = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \bar{x} + \begin{bmatrix} 2e^{-3t} \\ 3te^{-3t} \end{bmatrix}$

Soln:

$$\begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = 0$$

$(1-r)(-7-r)+16=0$

$-7-r+7r+r^2+16=0$

$r^2+6r+9=0$

$(r+3)^2=0$

$r_1=r_2=-3$

$(A-rI)\bar{z}=\bar{0}$

When $r=-3$

$$\begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 - 4z_2 = 0$

$4z_1 - 4z_2 = 0$

$z_1 = z_2$

Let $z_1=1$. $z_2=1$.

$\overline{z_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\bar{x} = C_1 e^{r_1 t}\overline{z^1} + C_2(te^{r_1 t}\overline{z^1} + e^{r_1 t}\bar{p})$$

$$(A - rI)\bar{p} = \bar{z}$$

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$4P_1 - 4P_2 = 1$

Let $P_2 = 0$.

$P_1 = 1/4$

$$P = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$$

$$\bar{x} = C_1 e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2\left(te^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1/4 \\ 0 \end{bmatrix}\right)$$

Now, we want to find U_1 and U_2 in

$U_1 e^{r_1 t}\overline{z^1} + U_2(te^{r_1 t}\overline{z^1} + e^{r_1 t}\bar{p})$

$$U_1' e^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + U_2'\left(te^{-3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1/4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2e^{-3t} \\ 3te^{-3t} \end{bmatrix}$$

$$U_1' e^{-3t} + U_2' te^{-3t} + \frac{U_2' e^{-3t}}{4} = 2e^{-3t} \quad (1)$$

$$U_1' e^{-3t} + U_2' te^{-3t} = 3te^{-3t} \quad (2)$$

Do (1) − (2)

$$\frac{U_2' e^{-3t}}{4} = 2e^{-3t} - 3te^{-3t}$$

$U_2' = 8 - 12t$

$U_2 = \int 8 - 12t \, dt$

$\quad = 8t - 6t^2 + C_2$

$U_1' e^{-3t} + (8-12t)te^{-3t} = 3te^{-3t}$

$U_1' + 8t - 12t^2 = 3t$

$U_1' = 12t^2 - 5t$

$U_1 = \int 12t^2 - 5t \, dt$

$= 4t^3 - \frac{5t^2}{2} + C_1$

$U_1 e^{r_1 t}\bar{z} + U_2(te^{r_1 t}\bar{z} + e^{r_1 t}\bar{p})$

$= \left(4t^3 - \frac{5t^2}{2} + C_1\right) e^{-3t}\begin{bmatrix}1\\1\end{bmatrix} +$

$\left(8t - 6t^2 + C_2\right)\left(te^{-3t}\begin{bmatrix}1\\1\end{bmatrix} + e^{-3t}\begin{bmatrix}1/4\\0\end{bmatrix}\right)$

E.g. Solve $\bar{x}' = \begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}\bar{x} + \begin{bmatrix}e^t\\t\end{bmatrix}$

Soln:

$\begin{vmatrix}2-r & -1\\3 & -2-r\end{vmatrix} = 0$

$(2-r)(-2-r)+3=0$

$-4-2r+2r+r^2+3=0$

$r^2-1=0$

$r^2=1$

$r_1=1, \; r_2=-1$

$(A-rI)\bar{z}=0$

when $r=1$

$\begin{bmatrix}1 & -1\\3 & -3\end{bmatrix}\begin{bmatrix}z_1\\z_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$z_1 - z_2 = 0$

$3z_1 - 3z_2 = 0$

$z_1 = z_2$

let $z_1 = 1$. $z_2 = 1$.

$$\bar{z^1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When $r = -1$

$$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - z_2 = 0$

$3z_1 - z_2 = 0$

$3z_1 = z_2$

let $z_1 = 1$. $z_2 = 3$

$$\bar{z^2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\left.\begin{aligned} \bar{x} &= C_1 e^{r_1 t} \bar{z^1} + C_2 e^{r_2 t} \bar{z^2} \\ &= C_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}\right\}$$ Gen soln of Homogeneous Linear System

Now, we want to solve for U_1 and U_2.

$U_1' e^{r_1 t} \bar{z^1} + U_2' e^{r_2 t} \bar{z^2} = \bar{g}$

We have:

$$U_1' e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + U_2' e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} e^{t} \\ t \end{bmatrix}$$

$U_1' e^{t} + U_2' e^{-t} = e^{t}$ (1)

$U_1' e^{t} + 3U_2' e^{-t} = t$ (2)

Do (1) − (2)

$-2U_2' e^{-t} = e^{t} - t$

$U_2' = -\frac{1}{2}\left(e^{2t} - te^{t}\right)$

$U_2 = \int -\frac{1}{2}\left(e^{2t} - te^{t}\right) dt$

$= -\frac{1}{2}\left(\frac{e^{2t}}{2} - \left(te^{t} - e^{t}\right)\right) + c_2$

$= \frac{-e^{2t}}{4} + \frac{te^{t}}{2} - \frac{e^{t}}{2}$

$U_1' = 1 - U_2' e^{-2t}$

$= 1 + \left(\frac{1}{2} - \frac{1}{2}te^{-t}\right)$

$= \frac{3}{2} - \frac{1}{2}te^{-t}$

$U_1 = \int \frac{3}{2} - \frac{1}{2}te^{-t}\ dt$

$= \frac{3t}{2} + \frac{(t+1)e^{-t}}{2}$

The particular soln is $U_1 e^{t}\begin{bmatrix}1\\1\end{bmatrix} + U_2 e^{-t}\begin{bmatrix}1\\3\end{bmatrix}$

E.g. Solve $\bar{x}' = \begin{bmatrix}2 & -5\\1 & -2\end{bmatrix}\bar{x} + \begin{bmatrix}-\cos(t)\\ \sin(t)\end{bmatrix}$

Soln:

$\begin{vmatrix}2-r & -5\\1 & -2-r\end{vmatrix} = 0$

$(2-r)(-2-r) + 5 = 0$

$-4 - 2r + 2r + r^2 + 5 = 0$

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$(A - rI)\bar{z} = \bar{0}$

Take $r = i$

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(2-i)z_1 - 5z_2 = 0$

$z_1 + (-2-i)z_2 = 0$

$(2-i)z_1 = 5z_2$

$$\frac{(2-i)z_1}{5} = z_2$$

Let $z_1 = 5.$ $z_2 = 2-i$

$$\overline{z^1} = \begin{bmatrix} 5 \\ 2-i \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$e^{r_1 t}\,\overline{z^1} = e^{it}\,\overline{z^1}$$

$$= (\cos(t) + i\sin(t))\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

$$= \cos(t) \begin{bmatrix} 5 \\ 2 \end{bmatrix} - \sin(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} +$$

$$i\left(\cos(t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(t) \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right)$$

The general soln to the H linear system is

$$c_1\left(\cos(t)\begin{bmatrix}5\\2\end{bmatrix} - \sin(t)\begin{bmatrix}0\\-1\end{bmatrix}\right) +$$

$$c_2\left(\cos(t)\begin{bmatrix}0\\-1\end{bmatrix} + \sin(t)\begin{bmatrix}5\\2\end{bmatrix}\right).$$

We can use variation of parameters to find the particular soln.

$$u_1'\left(\cos(t)\begin{bmatrix}5\\2\end{bmatrix} - \sin(t)\begin{bmatrix}0\\-1\end{bmatrix}\right) +$$

$$u_2'\left(\cos(t)\begin{bmatrix}0\\-1\end{bmatrix} + \sin(t)\begin{bmatrix}5\\2\end{bmatrix}\right) = \begin{bmatrix}-\cos(t)\\ \sin(t)\end{bmatrix}$$

$$u_1'(5\cos(t)) + u_2'(5\sin(t)) = -\cos(t)$$
$$u_1'(2\cos(t) + \sin(t)) + u_2'(-\cos(t) + 2\sin(t)) = \sin(t)$$

$$u_1' = \frac{-\cos(t) - 5u_2'\sin(t)}{5\cos(t)}$$

$$\left(\frac{-\cos(t) - 5u_2'\sin(t)}{5\cos(t)}\right)(2\cos(t) + \sin(t)) +$$

$$u_2'(-\cos(t) + 2\sin(t)) = \sin(t)$$

$$(-\cos(t) - 5u_2'\sin(t))(2\cos(t) + \sin(t)) +$$
$$u_2'(-\cos(t) + 2\sin(t))(5\cos(t)) = 5\sin(t)\cos(t)$$

$$-2\cos^2(t) - \sin(t)\cos(t) - 10\cancel{U_2'}\sin(t)\cos(t) -$$
$$5U_2'\sin^2(t) + 10U_2'\cancel{\sin}(t)\cos(t) - 5U_2'\cos^2(t)$$
$$= 5\sin(t)\cos(t)$$

$$-U_2'(5\sin^2(t) + 5\cos^2(t)) = 5\sin(t)\cos(t) + \sin(t)\cos(t) - 2\cos^2(t)$$

$$-5U_2' = 6\sin(t)\cos(t) - 2\cos^2(t)$$
$$U_2' = \frac{-1}{5}\left(6\sin(t)\cos(t) - 2\cos^2(t)\right)$$

$$U_2 = \int \frac{-1}{5}\left(6\sin(t)\cos(t) - 2\cos^2(t)\right)dt$$
$$= -\frac{1}{5}\left(3\cos^2(t) + t + \frac{\sin(2t)}{2}\right)$$

$$U_1' = \frac{-\cos(t) - 5U_2'\sin(t)}{5\cos(t)}$$
$$= \frac{-\cos(t) - 5\sin(t)\left(-\frac{1}{5}\left(6\sin(t)\cos(t) - 2\cos^2(t)\right)\right)}{5\cos(t)}$$
$$= -\frac{1}{5} + \frac{6}{5}\sin^2(t) + \frac{2}{5}\sin(t)\cos(t)$$

$$U_1 = \int -\frac{1}{5} + \frac{6}{5}\sin^2(t) + \frac{2}{5}\sin(t)\cos(t)\,dt$$
$$= \frac{-t}{5} - \frac{3(\sin(2t) - 2t)}{10} + \frac{\sin^2(t)}{5}$$

The particular soln is $U_1\left(\cos(t)\begin{bmatrix}5\\2\end{bmatrix} - \sin(t)\begin{bmatrix}0\\-1\end{bmatrix}\right)$

$+ U_2\left(\cos(t)\begin{bmatrix}0\\1\end{bmatrix} + \sin(t)\begin{bmatrix}5\\2\end{bmatrix}\right)$

E.g. Solve $\overline{x}' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \overline{x} + \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$

Soln:

$$\begin{vmatrix} 1-r & 1 \\ 4 & -2-r \end{vmatrix} = 0$$

$(1-r)(-2-r) - 4 = 0$

$-2 - r + 2r + r^2 - 4 = 0$

$r^2 + r - 6 = 0$

$(r+3)(r-2) = 0$

$r_1 = -3, \; r_2 = 2$

$(A - rI)\overline{z} = \overline{0}$

when $r = -3$

$$\begin{bmatrix} 1+3 & 1 \\ 4 & -2+3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 + z_2 = 0$

$4z_1 + z_2 = 0$

$4z_1 = -z_2$

Let $z_1 = 1. \; z_2 = -4$

$$\overline{z'} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

When $r=2$

$$\begin{bmatrix} 1-2 & 1 \\ 4 & -2-2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-z_1 + z_2 = 0$

$4z_1 + (-4)z_2 = 0$

$-z_1 = -z_2$

$z_1 = z_2$

Let $z_1 = 1$. $z_2 = 1$.

$$\overline{z^2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The general soln is $C_1 e^{r_1 t} \overline{z^1} + C_2 e^{r_2 t} \overline{z^2}$

which is $C_1 e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$U_1' e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + U_2' e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$$

$U_1' e^{-3t} + U_2' e^{2t} = e^{-2t}$ (1)

$-4U_1' e^{-3t} + U_2' e^{2t} = -2e^{t}$ (2)

Do (1) − (2)

$5U_1' e^{-3t} = e^{-2t} + 2e^{t}$

$$U_1' = \frac{1}{5}\left(e^{t} + 2e^{4t}\right)$$

$$U_1 = \int \frac{1}{5}\left(e^{t} + 2e^{4t}\right) dt$$

$$= \frac{e^{t}}{5} + \frac{1}{10} e^{4t}$$

$$U_2' e^{2t} = -2e^{t} + 4U_1' e^{-3t}$$

$$= -2e^{t} + \frac{4}{5}\left(e^{-2t} + 2e^{t}\right)$$

$$U_2' = -2e^{-t} + \frac{4}{5}e^{-4t} + \frac{8}{5}e^{-t}$$

$$= \frac{4}{5}e^{-4t} - \frac{2}{5}e^{-t}$$

$$U_2 = \int \frac{4e^{-4t}}{5} - \frac{2e^{-t}}{5}\,dt$$

$$= \frac{2e^{-t}}{5} - \frac{e^{-4t}}{5}$$

The particular soln is $U_1 e^{-3t}\begin{bmatrix}1\\-4\end{bmatrix} + U_2 e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$.

E.g. Solve $\bar{x}' = \begin{bmatrix}8 & -1\\63 & -8\end{bmatrix}\bar{x} + \begin{bmatrix}e^{t}\\t\end{bmatrix}$

Soln:

$$\begin{vmatrix}8-r & -1\\63 & -8-r\end{vmatrix} = 0$$

$$(8-r)(-8-r) + 63 = 0$$

$$-64 - 8r + 8r + r^2 + 63 = 0$$

$$r^2 - 1 = 0$$

$$r^2 = 1$$

$$r_1 = 1, \quad r_2 = -1$$

$(A - rI)\bar{z} = \bar{0}$

when $r = 1$

$$\begin{bmatrix} 8-1 & -1 \\ 63 & -8-1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$7z_1 - z_2 = 0$

$63z_1 - 9z_2 = 0$

$7z_1 = z_2$

let $z_1 = 1$. $z_2 = 7$.

$$\bar{z}^1 = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

when $r = -1$

$$\begin{bmatrix} 9 & -1 \\ 63 & -7 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$9z_1 - z_2 = 0$

$63z_1 - 7z_2 = 0$

$9z_1 = z_2$

let $z_1 = 1$. $z_2 = 9$.

$$\bar{z}^2 = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

The general soln of the H linear system is $c_1e^{t}\begin{bmatrix} 1 \\ 7 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 1 \\ 9 \end{bmatrix}$.

$$U_1' e^t \begin{bmatrix} 1 \\ 7 \end{bmatrix} + U_2' e^{-t} \begin{bmatrix} 1 \\ 9 \end{bmatrix} = \begin{bmatrix} e^t \\ t \end{bmatrix}$$

$U_1' e^t + U_2' e^{-t} = e^t$
$7U_1' e^t + 9U_2' e^{-t} = t$

$U_1' e^t = e^t - U_2' e^{-t}$
$U_1' = 1 - U_2' e^{-2t}$

$7(1 - U_2' e^{-2t}) e^t + 9U_2' e^{-t} = t$
$7(e^t - U_2' e^{-t}) + 9U_2' e^{-t} = t$
$7e^t - 7U_2' e^{-t} + 9U_2' e^{-t} = t$
$2U_2' e^{-t} = t - 7e^t$

$$U_2' = \frac{1}{2}\left(te^t - 7e^{2t}\right)$$

$$U_2 = \int \frac{1}{2}\left(te^t - 7e^{2t}\right) dt$$

$$= \frac{1}{2}\left(te^t - e^t - \frac{7}{2} e^{2t}\right)$$

$$U_1' = 1 - \left(\frac{1}{2}\left(te^t - 7e^{2t}\right)\right) e^{-2t}$$

$$= 1 - \tfrac{1}{2} te^{-t} + \tfrac{7}{2}$$

$$= \tfrac{9}{2} - \tfrac{1}{2} te^{-t}$$

$$U_1 = \int \frac{9}{2} - \frac{1}{2} te^{-t} \, dt$$

$$= \frac{9t}{2} - \frac{1}{2}\left(-te^{-t} - e^{-t}\right)$$

The particular soln is $U_1 e^t \begin{bmatrix} 1 \\ 7 \end{bmatrix} + U_2 e^{-t} \begin{bmatrix} 1 \\ 9 \end{bmatrix}$.

E.g. Solve $\bar{x}' = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\bar{x} + \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}$

Soln:

$\begin{vmatrix} 4-r & -2 \\ 8 & -4-r \end{vmatrix} = 0$

$(4-r)(-4-r)+16=0$

$-16-4r+4r+r^2+16=0$

$r^2=0$

$r_1=r_2=0$

$(A-rI)\bar{z}=\bar{0}$

when $r=0$

$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4z_1 - 2z_2 = 0$

$8z_1 - 4z_2 = 0$

$4z_1 = 2z_2$

$2z_1 = z_2$

Let $z_1=1.$ $z_2=2$

$\bar{z}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$(A-rI)\bar{p}=\bar{z}$

$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$4P_1 - 2P_2 = 1$

$8P_1 - 4P_2 = 2$

Let $P_2 = 0$.
$P_1 = \frac{1}{4}$

The general soln of the H linear system is
$C_1 e^{r_1 t}\bar{z_1} + C_2(te^{r_1 t}\bar{z_1} + e^{r_1 t}\bar{p})$ or

$$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2\left(t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}\right).$$

$$U_1' \begin{bmatrix} 1 \\ 2 \end{bmatrix} + U_2'\left(t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} t^{-3} \\ -t^{-2} \end{bmatrix}$$

$U_1' + U_2't + \frac{1}{4}U_2' = t^{-3}$
$2U_1' + 2U_2't = -t^{-2}$

$U_1' = -\frac{1}{2}t^{-2} - U_2't$

$-\frac{1}{2}t^{-2} - U_2't + U_2't + \frac{1}{4}U_2' = t^{-3}$
$\frac{1}{4}U_2' = t^{-3} + \frac{1}{2}t^{-2}$
$U_2' = 4t^{-3} + 2t^{-2}$
$U_2 = \int 4t^{-3} + 2t^{-2}\,dt$
$= -2t^{-2} - 2t^{-1}$

$U_1' = -\frac{1}{2}t^{-2} - (4t^{-3} + 2t^{-2})t$
$= -\frac{1}{2}t^{-2} - 4t^{-2} - 2t^{-1}$
$= -\frac{9}{2}t^{-2} - 2t^{-1}$
$U_1 = \int -\frac{9}{2}t^{-2} - 2t^{-1}\,dt$
$= \frac{9}{2}t^{-1} - 2\ln|t|$

The particular soln is $U_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + U_2\left(t\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}\right)$.

E.g. Solve $\bar{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \bar{x} + \begin{bmatrix} 2e^t \\ -e^t \end{bmatrix}$

Soln:

$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0$

$(1-r)^2 - 4 = 0$

$1 - 2r + r^2 - 4 = 0$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r_1 = 3, \quad r_2 = -1$

$(A - rI)\bar{z} = \bar{0}$

when $r = 3$

$\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2z_1 + z_2 = 0$

$4z_1 - 2z_2 = 0$

$-2z_1 = -z_2$

$2z_1 = z_2$

Let $z_1 = 1. \; z_2 = 2$

$\overline{z1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

When $r = -1$

$$\begin{bmatrix} 1+1 & 1 \\ 4 & 1+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2z_1 + z_2 = 0$
$4z_1 + 2z_2 = 0$

$2z_1 = -z_2$
Let $z_1 = 1$. $z_2 = -2$.

$$\overline{z^2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The general soln to the H linear system is
$C_1 e^{r_1 t}\overline{z^1} + C_2 e^{r_2 t}\overline{z^2}$ or $C_1 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$$U_1' e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + U_2' e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}$$

$U_1' e^{3t} + U_2' e^{-t} = 2e^{t}$ (1)
$2U_1' e^{3t} - 2U_2' e^{-t} = -e^{-t}$ (2)

Do 2. (1) − (2)

$4U_2' e^{-t} = 4e^{t} + e^{-t}$
$U_2' = e^{2t} + \frac{1}{4}$
$U_2 = \int e^{2t} + \frac{1}{4}\, dt$
$= \frac{e^{2t}}{2} + \frac{t}{4}$

$$U_1' e^{3t} = 2e^t - U_2' e^{-t}$$
$$= 2e^t - (e^{2t} + \tfrac{1}{4})e^{-t}$$
$$= 2e^t - e^t - e^{-t}/4$$
$$U_1' = 2e^{-2t} - e^{-2t} - \frac{e^{-4t}}{4}$$
$$= e^{-2t} - \frac{e^{-4t}}{4}$$

$$U_1 = \int e^{-2t} - \frac{e^{-4t}}{4}\, dt$$
$$= \frac{e^{-2t}}{-2} + \frac{e^{-4t}}{16}$$

The particular soln is $U_1 e^{3t}\begin{bmatrix}1\\2\end{bmatrix} + U_2 e^{-t}\begin{bmatrix}1\\-2\end{bmatrix}$.

E.g. Solve $\bar{x}' = \begin{bmatrix}2 & -1\\3 & -2\end{bmatrix}\bar{x} + \begin{bmatrix}e^t\\-e^t\end{bmatrix}$

Soln:

$$\begin{vmatrix}2-r & -1\\3 & -2-r\end{vmatrix} = 0$$
$$(2-r)(-2-r) + 3 = 0$$
$$-4 - 2r + 2r + r^2 + 3 = 0$$
$$r^2 - 1 = 0$$
$$r^2 = 1$$
$$r_1 = 1,\ r_2 = -1$$

$(A - rI)\bar{z} = \bar{0}$

when $r=1$

$$\begin{bmatrix} 2-1 & -1 \\ 3 & -2-1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z_1 - z_2 = 0$
$3z_1 - 3z_2 = 0$

$z_1 = z_2$
let $z_1 = 1$. $z_2 = 1$.

$$\overline{z^1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

when $r = -1$

$$\begin{bmatrix} 2+1 & -1 \\ 3 & -2+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3z_1 - z_2 = 0$
$3z_1 - z_2 = 0$

$3z_1 = z_2$
let $z_1 = 1$. $z_2 = 3$

$$\overline{z^2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The general soln of the H linear system is
$c_1e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$U_1' e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + U_2' e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \end{bmatrix}$$

$U_1' e^t + U_2' e^{-t} = e^t$ (1)
$U_1' e^t + 3U_2' e^{-t} = -e^t$ (2)

Do (1) − (2)
$-2U_2' e^{-t} = 2e^t$
$U_2' = -e^{2t}$
$U_2 = \int -e^{2t}\, dt$
$= \dfrac{-e^{2t}}{2}$

$U_1' e^t = e^t - U_2' e^{-t}$
$= e^t - e^{-t}(-e^{2t})$
$= e^t + e^t$
$= 2e^t$
$U_1' = 2$
$U_1 = \int 2\, dt$
$= 2t$

The particular soln is $U_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + U_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

E.g. Solve $\bar{x}' = \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} e^{-3t} \\ 2e^{-4t} \end{bmatrix}$

Soln:

$\begin{vmatrix} -6-r & 4 \\ -1 & -2-r \end{vmatrix} = 0$

$(-6-r)(-2-r)+4=0$

$12+6r+2r+r^2+4=0$

$r^2+8r+16=0$

$(r+4)^2=0$

$r_1=r_2=-4$

$(A-rI)\bar{z}=\bar{0}$

When $r=-4$

$\begin{bmatrix} -6+4 & 4 \\ -1 & -2+4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2z_1 + 4z_2 = 0$

$-z_1 + 2z_2 = 0$

$-z_1 = -2z_2$

$\frac{z_1}{2} = z_2$

Let $z_1=2.$ $z_2=1.$

$\bar{z_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$(A - rI)\bar{p} = \bar{z}$

$$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$-2P_1 + 4P_2 = 2$
$-P_1 + 2P_2 = 1$

let $P_2 = 0$.
$P_1 = -1$

$$\bar{p} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The general soln to the H linear system is $C_1 e^{r_1 t} \bar{z_1} + C_2 (t e^{r_1 t} \bar{z_1} + e^{r_1 t} \bar{p})$ or

$$C_1 e^{-4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left(t e^{-4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-4t} \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right).$$

$$U_1' e^{-4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + U_2' \left(t e^{-4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-4t} \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} e^{-3t} \\ 2e^{-4t} \end{bmatrix}$$

$2U_1' e^{-4t} + 2U_2' t e^{-4t} - U_2' e^{-4t} = e^{-3t}$
$U_1' e^{-4t} + U_2' t e^{-4t} = 2e^{-4t}$

$U_1' + U_2' t = 2$
$U_1' = 2 - U_2' t$

$2(2-U_2't)e^{-4t} + 2U_2'te^{-4t} - U_2'e^{-4t} = e^{-3t}$

$4e^{-4t} - 2U_2'\cancel{t}e^{-4t} + 2U_2'\cancel{t}e^{-4t} - U_2'e^{-4t} = e^{-3t}$

$4 - U_2' = e^{t}$

$-U_2' = e^{t} - 4$

$U_2' = 4 - e^{t}$

$U_2 = \int 4 - e^{t} \, dt$

$= 4t - e^{t}$

$U_1' = 2 - U_2't$

$= 2 - (4-e^{t})t$

$= 2 - (4t - te^{t})$

$= 2 - 4t + te^{t}$

$U_1 = \int 2 - 4t + te^{t} \, dt$

$= 2t - 2t^2 + te^{t} - e^{t}$

The particular soln is $U_1 e^{-4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} +$

$U_2 \left(te^{-4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-4t} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right).$

E.g. Solve $\bar{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \bar{x} + \begin{bmatrix} e^{-t} \cos(t) \\ e^{-t} \sin(t) \end{bmatrix}$

Soln:

$\begin{vmatrix} -1-r & -4 \\ 1 & -1-r \end{vmatrix} = 0$

$(-1-r)^2 + 4 = 0$

$r^2 + 2r + 1 + 4 = 0$

$r^2 + 2r + 5 = 0$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-2 \pm \sqrt{4-20}}{2}$$
$$= \frac{-2 \pm \sqrt{-16}}{2}$$
$$= \frac{-2 \pm 4i}{2}$$
$$= -1 \pm 2i$$

$(A - rI)\bar{z} = \bar{0}$

when $r = -1+2i$

$$\begin{bmatrix} -1-(-1+2i) & -4 \\ 1 & -1-(-1+2i) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-2i)z_1 - 4z_2 = 0$

$z_1 + (-2i)z_2 = 0$

$-2iz_1 = 4z_2$

$\frac{-iz_1}{2} = z_2$

let $z_1 = 2.$ $z_2 = -i$

$$\bar{z}' = \begin{bmatrix} 2 \\ -i \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$e^{r_1 t}\overline{z^1} = e^{(-1+2i)t}\overline{z^1}$$
$$= e^{-t}\cdot e^{2it}\cdot\overline{z^1}$$
$$= e^{-t}(\cos(2t)+i\sin(2t))\left(\begin{bmatrix}2\\0\end{bmatrix}+i\begin{bmatrix}0\\-1\end{bmatrix}\right)$$
$$= e^{-t}\left(\cos(2t)\begin{bmatrix}2\\0\end{bmatrix}-\sin(2t)\begin{bmatrix}0\\-1\end{bmatrix}\right)+$$
$$ie^{-t}\left(\cos(2t)\begin{bmatrix}0\\-1\end{bmatrix}+\sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right)$$

The general soln to the H linear system is

$$C_1e^{-t}\left(\cos(2t)\begin{bmatrix}2\\0\end{bmatrix}-\sin(2t)\begin{bmatrix}0\\-1\end{bmatrix}\right)+$$
$$C_2e^{-t}\left(\cos(2t)\begin{bmatrix}0\\-1\end{bmatrix}+\sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right)$$

$$U_1'e^{-t}\left(\cos(2t)\begin{bmatrix}2\\0\end{bmatrix}-\sin(2t)\begin{bmatrix}0\\-1\end{bmatrix}\right)+$$
$$U_2'e^{-t}\left(\cos(2t)\begin{bmatrix}0\\-1\end{bmatrix}+\sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right)=\begin{bmatrix}e^{-t}\cos(t)\\e^{-t}\sin(t)\end{bmatrix}$$

$$U_1'e^{-t}(2\cos(2t)) + U_2'e^{-t}(2\sin(2t)) = e^{-t}\cos(t)$$
$$U_1'e^{-t}\sin(2t) - U_2'e^{-t}\cos(2t) = e^{-t}\sin(t)$$

$$U_1'(2\cos(2t)) + U_2'(2\sin(2t)) = \cos(t) \quad (1)$$
$$U_1'\sin(2t) - U_2'\cos(2t) = \sin(t) \quad (2)$$

Multiply (1) by $\cos(2t)$.
Multiply (2) by $2\sin(2t)$.
Do (1) + (2)

$$U_1'(2\cos^2(2t)) + U_2'(2\sin(2t)\cos(2t)) +$$
$$U_1'(2\sin^2(2t)) - U_2'(2\sin(2t)\cos(2t)) =$$
$$\cos(t)\cdot\cos(2t) + 2\sin(t)\sin(2t).$$

$$2U_1' = \cos(t)\cos(2t) + 2\sin(t)\sin(2t)$$
$$U_1 = \frac{1}{2}\int \cos(t)\cos(2t) + 2\sin(t)\sin(2t)\,dt$$

Plug U_1' into (2) to get U_2'.

$$\left(\frac{\cos(t)\cos(2t) + 2\sin(t)\sin(2t)}{2}\right) - U_2'\cos(2t) = \sin(t)$$

$$-U_2'\cos(2t) = \sin(t) - \left(\frac{\cos(t)\cos(2t) + 2\sin(t)\sin(2t)}{2}\right)$$

$$U_2' = \frac{-1}{\cos(2t)}\left(\sin(t) - \left(\frac{\cos(t)\cos(2t) + 2\sin(t)\sin(2t)}{2}\right)\right)$$

$$U_2 = \int \uparrow$$

The particular soln is $U_1 e^{-t}\left(\cos(2t)\begin{bmatrix}2\\0\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\-1\end{bmatrix}\right) + U_2' e^{-t}\left(\cos(2t)\begin{bmatrix}0\\-1\end{bmatrix} + \sin(2t)\begin{bmatrix}2\\0\end{bmatrix}\right)$

3 x 3 Linear Systems

- We will only be dealing with H linear systems for 3 by 3 coefficient matrices.
- Superposition of Homogeneous System:

If $\overline{x^1}, \overline{x^2}, \ldots, \overline{x^n}$ are solns, then $C_1\overline{x^1} \pm C_2\overline{x^2} \pm \ldots \pm C_n\overline{x^n}$ is also a soln.

For 2x2: If $\overline{x^1}$ and $\overline{x^2}$ are solns, then $C_1\overline{x^1} \pm C_2\overline{x^2}$ is also a soln.

For 3x3: If $\overline{x^1}$, $\overline{x^2}$ and $\overline{x^3}$ are solns, then $C_1\overline{x^1} \pm C_2\overline{x^2} \pm C_3\overline{x^3}$ is also a soln.

- Let $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \bar{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are functions of t.

If $c_1\bar{x} + c_2\bar{y} + c_3\bar{z} = 0$ for all t, then they are linearly independent.

To know if $\bar{x}, \bar{y}$, and $\bar{z}$ are linearly dependent or not, we can use the Wronksian.

$$w = \begin{vmatrix} x_1(t) & y_1(t) & z_1(t) \\ x_2(t) & y_2(t) & z_2(t) \\ x_3(t) & y_3(t) & z_3(t) \end{vmatrix}$$

$\bar{x}$, $\bar{y}$ and $\bar{z}$ are linearly independent iff $w[x, y, z] = 0$.

- The general soln for H systems for a 3x3 coefficient matrix is $c_1\overline{x^1}(t) + c_2\overline{x^2}(t) + c_3\overline{x^3}(t)$, where $\overline{x^1}$, $\overline{x^2}$ and $\overline{x^3}$ are linearly independent solns.

E.g. Solve $\bar{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 0-r & 1 & 1 \\ 1 & 0-r & 1 \\ 1 & 1 & 0-r \end{vmatrix} = 0$$

$(-r)(r^2-1) - [-r-1] + [1-(-r)] = 0$

$-r^3 + r + r + 1 + 1 + r = 0$

$-r^3 + 3r + 2 = 0$

when $r=-1 \rightarrow -(-1)^3 + 3(-1) + 2 = 1-3+2 = 0$

$r=-1$ is a root

$$\begin{array}{r}
-r^2 + r + 2 \\
r+1 \overline{\big) -r^3 + 0r^2 + 3r + 2} \\
-(\underline{-r^3 - r^2)} \\
r^2 + 3r \\
-\underline{(r^2 + r)} \\
2r + 2 \\
-\underline{(2r + 2)} \\
0
\end{array}$$

$(r+1)(-r^2+r+2) = -r^3+3r+2$

To factor $-r^2+r+2$, I'll use the quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-1 \pm \sqrt{1-4(-1)(2)}}{-2}$$
$$= \frac{-1 \pm 3}{-2}$$
$$= -1 \text{ or } 2$$

Hence, $r_1 = r_2 = -1$, $r_3 = 2$

$(A - rI)\bar{z} = \bar{0}$
when $r = -1$

$$\begin{bmatrix} 0+1 & 1 & 1 \\ 1 & 0+1 & 1 \\ 1 & 1 & 0+1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} z_1 + z_2 + z_3 = 0 \\ z_1 + z_2 + z_3 = 0 \\ z_1 + z_2 + z_3 = 0 \end{array}\right\} \text{Redundant}$$

$z_1 = -z_2 - z_3$
Let $z_2 = 0$ and $z_3 = 1$. $z_1 = -1$
Let $z_2 = 1$ and $z_3 = 0$. $z_1 = -1$

$$\bar{z}^1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{z}^2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

when $r=2$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2z_1 + z_2 + z_3 = 0$ (1)
$z_1 - 2z_2 + z_3 = 0$ (2)
$z_1 + z_2 - 2z_3 = 0$ (3)

If you do (2) − (3), eqn (2) becomes $-3z_2 + 3z_3 = 0$.
If you do $2 \cdot$(3) + (1), eqn (3) becomes $3z_2 - 3z_3 = 0$.

Now, we have
$-2z_1 + z_2 + z_3 = 0$
$-3z_2 + 3z_3 = 0$
$3z_2 - 3z_3 = 0$ $\leftarrow$ Redundant

$-3z_2 = -3z_3$
$z_2 = z_3$
Let $z_2 = 1$. $z_3 = 1$.

$-2z_1 + 2 = 0$
$-2z_1 = -2$
$z_1 = 1$

$$\overline{z^3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The general soln is
$$\overline{x}' = c_1 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & -1 & -1 \\ 1 & 1-r & 0 \\ 3 & 0 & 1-r \end{vmatrix} = 0$$

$$(1-r)\left[(1-r)^2 - 0\right] - (-1)\left[(1-r) - 0\right] + (-1)\left[0 - 3(1-r)\right] = 0$$
$$(1-r)^3 + (1-r) + 3(1-r) = 0$$
$$(1-r)^3 + 4(1-r) = 0$$
$$(1-r)\left[(1-r)^2 + 4\right] = 0$$
$$(1-r)\left[r^2 - 2r + 1 + 4\right] = 0$$

To factor $r^2 - 2r + 4$, I'll use the quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{2 \pm \sqrt{4 - 20}}{2}$$
$$= \frac{2 \pm 4i}{2}$$
$$= 1 \pm 2i$$

$r_1 = 1, \quad r_2 = 1 + 2i$

$(A - rI)\bar{z} = \bar{0}$

when $r = 1$

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-z_2 - z_3 = 0$

$z_1 = 0$

$3z_1 = 0$ ← Redundant

$-z_2 = z_3$

at $z_2 = 1$. $z_3 = -1$

$$\bar{z^1} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

when $r = 1 + 2i$

$$\begin{bmatrix} -2i & -1 & -1 \\ 1 & -2i & 0 \\ 3 & 0 & -2i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2iz_1 - z_2 - z_3 = 0$ (1)

$z_1 - 2iz_2 = 0$ (2)

$3z_1 - 2iz_3 = 0$ (3)

If you do $2i \cdot (1) + (2)$, we get (3).

$z_1 = 2i z_2 \rightarrow$ let $z_1 = 2i$. $z_2 = 1$
$3z_1 = 2i z_3 \rightarrow$ let $z_1 = 2i$. $z_3 = 3$

$$\overline{z^2} = \begin{bmatrix} 2i \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} e^{r_2 t}\overline{z^2} &= e^{(1+2i)t}\overline{z^2} \\ &= e^t \cdot e^{(2t)i} \cdot \overline{z^2} \\ &= e^t(\cos(2t) + i\sin(2t))\left(\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) \\ &= e^t\left(\cos(2t)\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} - \sin(2t)\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) + \\ &\quad ie^t\left(\cos(2t)\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \sin(2t)\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right) \end{aligned}$$

The general soln is $\bar{x} = C_1 e^t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} +$

$$C_2 e^t\left(\cos(2t)\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} - \sin(2t)\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right) +$$

$$C_3 e^t\left(\cos(2t)\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \sin(2t)\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right)$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{vmatrix} = 0$$

$(1-r)\left[(2-r)(1-r)-1\right] - \left[1-r-2\right] + 2\left[1-2(2-r)\right] = 0$

$(1-r)\left[2-2r-r+r^2-1\right] - \left[-r-1\right] + 2\left[1-4+2r\right] = 0$

$(1-r)\left[r^2-3r+1\right] + r+1-6+4r = 0$

$r^2-3r+1-r^3+3r^2-r+r+1-6+4r = 0$

$-r^3 + 4r^2 + r - 4 = 0$

$-r^2(r-4)+(r-4) = 0$

$(r-4)(-r^2+1) = 0$

$-r^2+1=0$

$-r^2=-1$

$r^2=1$

$r=\pm 1$

$r_1=4,\ r_2=1,\ r_3=-1$

$(A-rI)\bar{z} = \bar{0}$

when $r=4$

$$\begin{bmatrix} 1-4 & 1 & 2 \\ 1 & 2-4 & 1 \\ 2 & 1 & 1-4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-3z_1 + z_2 + 2z_3 = 0$ (1)

$z_1 - 2z_2 + z_3 = 0$ (2)

$2z_1 + z_2 - 3z_3 = 0$ (3)

If you do -1((2)+(3)), you get (1).
Hence, (1) is redundant.

$z_2 = -2z_1 + 3z_3$
$z_1 - 2(-2z_1 + 3z_3) + z_3 = 0$
$z_1 + 4z_1 - 6z_3 + z_3 = 0$
$5z_1 - 5z_3 = 0$
$z_1 = z_3$
Let $z_1 = 1$. $z_3 = 1$. $z_2 = 1$.

$$\overline{z'} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

When r = 1

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$z_2 + 2z_3 = 0$ (1)
$z_1 + z_2 + z_3 = 0$ (2)
$2z_1 + z_2 = 0$ (3)

If you do 2·(2) - (3), you get (1).
Hence, (1) is redundant.

$2z_1 = -z_2$
$z_1 - 2z_1 + z_3 = 0$
$-z_1 + z_3 = 0$
$z_1 = z_3$

Let $z_1=1.$ $z_2=-2,$ $z_3=1.$

$$\overline{z^2}=\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

when $r=-1$

$$\begin{bmatrix}2&1&2\\1&3&1\\2&1&2\end{bmatrix}\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$2z_1+z_2+2z_3=0$

$z_1+3z_2+z_3=0$

$2z_1+z_2+2z_3=0$ ← Redundant

$z_1=-3z_2-z_3$

$2(-3z_2-z_3)+z_2+2z_3=0$

$-6z_2-2z_3+z_2+2z_3=0$

$-5z_2=0$

$z_2=0$

$z_1=-z_3$

Let $z_1=1.$ $z_3=-1.$

$$\overline{z^3}=\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

The general soln is $c_1e^{4t}\begin{bmatrix}1\\1\\1\end{bmatrix}+$

$c_2e^{t}\begin{bmatrix}1\\-2\\1\end{bmatrix}+c_3e^{-t}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

E.g. Solve $\overline{x}' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \overline{x}$

Soln:

$$\begin{vmatrix} 3-r & 2 & 4 \\ 2 & 0-r & 2 \\ 4 & 2 & 3-r \end{vmatrix} = 0$$

$(3-r)\left[(-r)(3-r)-4\right]-2\left[2(3-r)-8\right]+4\left[4-(-4r)\right]=0$

$(3-r)\left[r^2-3r-4\right]-2\left[-2r+6-8\right]+4\left[4+4r\right]=0$

$-r^3+3r^2+4r+3r^2-9r-12+4r+4+16+16r=0$

$-r^3+6r^2+15r+8=0$

Take $r=-1$.

$-(-1)^3+6(-1)^2+15(-1)+8$

$=1+6-15+8$

$=0$

Hence, $r=-1$ is a root.

$$\begin{array}{r l}
 & -r^2+7r+8 \\ \hline
r+1 \; \big) & -r^3+6r^2+15r+8 \\
 & -(\underline{-r^3-r^2}) \\
 & \quad 7r^2+15r \\
 & \quad -(\underline{7r^2+7r}) \\
 & \qquad 8r+8 \\
 & \qquad -(\underline{8r+8}) \\
 & \qquad\quad 0
\end{array}$$

$(r+1)(-r^2+7r+8)=-r^3+6r^2+15r+8$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{49 + 32}}{-2}$$

$$= \frac{-7 \pm \sqrt{81}}{-2}$$

$$= \frac{-7 \pm 9}{2}$$

$$= 1 \text{ or } -8$$

The roots are $r=-1$ and $r=8$

$(A - rI)\bar{z} = \bar{0}$

when $r=-1$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4z_1 + 2z_2 + 4z_3 = 0$

$2z_1 + z_2 + 2z_3 = 0$ ← Redundant

$4z_1 + 2z_2 + 4z_3 = 0$ ← Redundant

$2z_1 + z_2 + 2z_3 = 0$

$z_2 = -2z_1 - 2z_3$

Let $z_1 = 0$ and $z_3 = 1$. $z_2 = -2$

$$\bar{z}^1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Let $z_1 = 1$ and $z_3 = 0$. $z_2 = -2$

$$\bar{z}^2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

When $r = 8$

$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5z_1 + 2z_2 + 4z_3 = 0$ (1)

$2z_1 - 8z_2 + 2z_3 = 0$ (2)

$4z_1 + 2z_2 - 5z_3 = 0$ (3)

If you do $(-2)[(1) + (3)]$, you get (2).
Hence, (2) is redundant.

$2z_2 = -4z_1 + 5z_3$

$-5z_1 - 4z_1 + 5z_3 + 4z_3 = 0$

$-9z_1 = -9z_3$

$z_1 = z_3$

Let $z_1 = 1$. $z_3 = 1$.

$z_2 = 1/2$

$$\bar{z}^3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

The general soln is $\bar{x} = C_1 e^{-t} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} +$

$$C_2 e^{-t} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + C_3 e^{-8t} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{vmatrix} = 0$$

$$(1-r)\left[(1-r)^2-1\right] - \left[2(1-r) - 0\right] + \left[-2-0\right] = 0$$

$$(1-r)^3 - 1 + r - 2 + 2r - 2 = 0$$

$$-r^3 + 3r^2 - 3r + 1 + 3r - 5 = 0$$

$$-r^3 + 3r^2 - 4 = 0$$

$r = -1$ is a root.

$$\begin{array}{r} -r^2 + 4r - 4 \\ r+1 \overline{\big)\, -r^3 + 3r^2 + 0r - 4} \\ -(\underline{-r^3 - r^2)} \end{array}$$

$$\begin{array}{r} 4r^2 + 0r \\ -(\underline{4r^2 + 4r)} \\ -4r - 4 \\ -(\underline{-4r - 4)} \\ 0 \end{array}$$

$(r+1)(-r^2+4r-4) = -r^3 + 3r^2 - 4$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{16 - 16}}{-2}$$

$$= 2$$

$r_1 = -1, \; r_2 = r_3 = 2$

$(A - rI)\bar{z} = \bar{0}$

when $r = -1$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2z_1 + z_2 + z_3 = 0$ (1)

$2z_1 + 2z_2 - z_3 = 0$ (2)

$-z_2 + 2z_3 = 0$ (3)

If you do (1) - (2), you get (3).
Hence, (3) is redundant.

$$2z_1 + z_2 + z_3 = 0$$
$$2z_1 + 2z_2 - z_3 = 0$$

$$2z_1 = -z_2 - z_3$$
$$-z_2 + 2z_2 - z_3 - z_3 = 0$$
$$z_2 = 2z_3$$

Let $z_3 = 1$. $z_2 = 2$. $z_1 = -3/2$

$$\bar{z}^1 = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$$

When $r = 2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-z_1 + z_2 + z_3 = 0 \quad (1)$$
$$2z_1 - z_2 - z_3 = 0 \quad (2)$$
$$-z_2 - z_3 = 0 \quad (3)$$

If you do $(-1)[2 \cdot (1) + (2)]$, you get (3), so (3) is redundant.

$$-z_1 + z_2 + z_3 = 0$$
$$2z_1 - z_2 - z_3 = 0 \rightarrow z_3 = 2z_1 - z_2$$

$-z_1 + z_2 + 2z_1 - z_2 = 0$
$z_1 = 0$

$z_3 = -z_2$
let $z_3 = 1.$ $z_2 = -1.$

$$\overline{z^2} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

To find $\overline{z^3}$, we can use the method of generalized eigenvector.

$\overline{x} = C_1 e^{r_1 t} \overline{z^1} + C_2 e^{r_2 t} \overline{z^2} + C_3 (t e^{r_i t} \overline{z^i} + e^{r_i t} \overline{p})$
where $i = 1$ or 2 and $\overline{p}$ is an unknown vector.

For this example, let $i = 2$.
$(A - rI)\overline{p} = \overline{z}$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$-P_1 + P_2 + P_3 = 0 \rightarrow -P_2 = -P_1 + P_3$
$2P_1 - P_2 - P_3 = -1$

$2P_1 - P_1 + P_3 - P_3 = -1$
$P_1 = -1$
let $P_2 = 0.$ $P_3 = 1$

$$\overline{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The general soln is $\bar{x} = C_1 e^{-t}\begin{bmatrix}-3\\4\\2\end{bmatrix} +$

$$C_2 e^{2t}\begin{bmatrix}0\\-1\\1\end{bmatrix} + C_3\left(te^{2t}\begin{bmatrix}0\\-1\\1\end{bmatrix} + e^{2t}\begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$$

E.g. Solve $\bar{x}' = \begin{bmatrix}1 & 1 & 4\\1 & 4 & 1\\4 & 1 & 1\end{bmatrix}\bar{x}$

Soln:

$$\begin{vmatrix}1-r & 1 & 4\\1 & 4-r & 1\\4 & 1 & 1-r\end{vmatrix} = 0$$

$(1-r)\left[(4-r)(1-r)-1\right] - (1)\left[(1-r)-4\right] + 4\left[1-4(4-r)\right] = 0$

$(1-r)(4-4r-r+r^2-1) - (-3-r) + 4(1-16+4r) = 0$

$(1-r)(r^2-5r+3) + 3+r + 16r - 60 = 0$

$r^2-5r+3-r^3+5r^2-3r+3+r+16r-60 = 0$

$-r^3+6r^2+9r-54 = 0$

$-r^2(r-6)+9(r-6) = 0$

$(r-6)(-r^2+9) = 0$

$(r-6)(r^2-9) = 0$

$(r-6)(r-3)(r+3) = 0$

$r_1 = 6,\ r_2 = 3,\ r_3 = -3$

$(A-rI)\bar{z}=\bar{0}$

when $r=6$

$$\begin{bmatrix} -5 & 1 & 4 \\ 1 & -2 & 1 \\ 4 & 1 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5z_1 + z_2 + 4z_3 = 0$ (1)

$z_1 - 2z_2 + z_3 = 0$ (2)

$4z_1 + z_2 - 5z_3 = 0$ (3)

If you do $(-1)((1)+(2))$, you get (3).
Hence, (3) is redundant.

$-z_2 = -5z_1 + 4z_3$

$z_1 + 2(-5z_1 + 4z_3) + z_3 = 0$

$z_1 - 10z_1 + 8z_3 + z_3 = 0$

$-9z_1 + 9z_3 = 0$

$z_1 = z_3$

Let $z_1 = 1$. $z_3 = 1$. $z_2 = 1$

$$\bar{z_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

when $r=3$

$$\begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2z_1 + z_2 + 4z_3 = 0 \quad (1)$$
$$z_1 + z_2 + z_3 = 0 \quad (2)$$
$$4z_1 + z_2 - 2z_3 = 0 \quad (3)$$

If you do $(-1)((1) - 2\cdot(2))$, you get (3).
Hence, (3) is redundant.

$-3z_1 + 3z_3 = 0$

$z_1 = z_3$

Let $z_1 = 1$. $z_3 = 1$.

$z_2 = -z_1 - z_3$

$= -2$

$$\overline{z^2} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

When $r = -3$

$$\begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4z_1 + z_2 + 4z_3 = 0$

$z_1 + 7z_2 + z_3 = 0$

$4z_1 + z_2 + 4z_3 = 0$ ← Redundant

$z_2 = -4z_1 - 4z_3$

$z_1 + 7(-4z_1 - 4z_3) + z_3 = 0$

$z_1 - 28z_1 - 28z_3 + z_3 = 0$

$-27z_1 - 27z_3 = 0$

$-27z_1 = 27z_3$

$z_1 = -z_3$

Let $z_1 = 1$. $z_3 = -1$.

$z_2 = 0$

$$\overline{z^3} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Hence, the general soln is $\bar{x} = C_1 e^{6t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} +$

$C_2 e^{3t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + C_3 e^{-3t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

E.g. Solve $\bar{x}' = \begin{bmatrix} 1 & 1 & 6 \\ 1 & 6 & 1 \\ 6 & 1 & 1 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & 1 & 6 \\ 1 & 6-r & 1 \\ 6 & 1 & 1-r \end{vmatrix} = 0$$

$(1-r)\left[(6-r)(1-r)-1\right] - \left[1-r-6\right] + 6\left[1-6(6-r)\right] = 0$

$(1-r)\left[6-6r-r+r^2-1\right] - \left[-5-r\right] + 6\left[1-36+6r\right] = 0$

$(1-r)\left[r^2-7r+5\right] + r+5+6\left[-35+6r\right] = 0$

$r^2-7r+5-r^3+7r^2-5r+r+5-210+36r = 0$

$-r^3+8r^2+25r-200 = 0$

$-r^2(r-8)+25(r-8) = 0$

$(r-8)(-r^2+25) = 0$

$(r-8)(r^2-25) = 0$

$(r-8)(r-5)(r+5) = 0$

$r_1 = 8, \; r_2 = 5, \; r_3 = -5$

$(A-rI)\bar{z} = \bar{0}$

when $r=8$

$$\begin{bmatrix} 1-8 & 1 & 6 \\ 1 & 6-8 & 1 \\ 6 & 1 & 1-8 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7z_1 + z_2 + 6z_3 = 0$ (1)

$z_1 - 2z_2 + z_3 = 0$ (2)

$6z_1 + z_2 - 7z_3 = 0$ (3)

If you do (-1)((1) + (3)), you get (2).
Hence, (2) is redundant.

$z_2 = 7z_1 - 6z_3$

$6z_1 + 7z_1 - 6z_3 - 7z_3 = 0$

$z_1 = z_3$

Lt $z_1 = 1$. $z_3 = 1$ $z_2 = 1$

$$\bar{z}^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

When $r = 5$

$$\begin{bmatrix} 1-5 & 1 & 6 \\ 1 & 6-5 & 1 \\ 6 & 1 & 1-5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4z_1 + z_2 + 6z_3 = 0$ (1)

$z_1 + z_2 + z_3 = 0$ (2)

$6z_1 + z_2 - 4z_3 = 0$ (3)

If you do (-1)((1) - 2(2)), you get (3).
Hence, (3) is redundant.

$z_2 = -z_1 - z_3$

$-4z_1 - z_1 - z_3 + 6z_3 = 0$

$-5z_1 + 5z_3 = 0$

$z_1 = z_3$

Lt $z_1 = 1$. $z_3 = 1$. $z_2 = -2$

$$\bar{z}^2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

When $r=-5$

$$\begin{bmatrix} 6 & 1 & 6 \\ 1 & 11 & 1 \\ 6 & 1 & 6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$6z_1 + z_2 + 6z_3 = 0$

$z_1 + 11z_2 + z_3 = 0$

$6z_1 + z_2 + 6z_3 = 0$ ← Redundant

$z_2 = -6z_1 - 6z_3$

$z_1 + 11(-6z_1 - 6z_3) + z_3 = 0$

$-65z_1 - 65z_3 = 0$

$z_1 = -z_3$

let $z_1 = 1$. $z_3 = -1$. $z_2 = 0$

$$\overline{z^3} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Hence, the general soln is $\bar{x} =$

$$C_1 e^{8t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + C_3 e^{-5t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & -1 & 4 \\ 3 & 2-r & -1 \\ 2 & 1 & -1-r \end{vmatrix} = 0$$

$$(1-r)\left[(2-r)(-1-r)+1\right]-(-1)\left[3(-1-r)+2\right]+4\left[3-2(2-r)\right]=0$$
$$(1-r)\left[-2-2r+r+r^2+1\right]+\left[-3-3r+2\right]+4\left[3-4+2r\right]=0$$
$$(1-r)\left[r^2-r-1\right]+\left[-3r-1\right]+4\left[2r-1\right]=0$$
$$r^2-r-1-r^3+r^2+r-3r-1+8r-4=0$$
$$-r^3+2r^2+5r-6=0$$

$r=1$ is a root

$$\begin{array}{r l}
 & -r^2 + r + 6 \\ \cline{2-2}
r-1 & \big) -r^3 + 2r^2 + 5r - 6 \\
 & -(\underline{-r^3 + r^2}) \\
 & \qquad r^2 + 5r \\
 & \qquad -(\underline{r^2 - r}) \\
 & \qquad\qquad 6r-6 \\
 & \qquad\qquad -(\underline{6r-6}) \\
 & \qquad\qquad\quad 0
\end{array}$$

$$(r-1)(-r^2+r+6)=0$$
$$(r-1)(r^2-r-6)=0$$
$$(r-1)(r-3)(r+2)=0$$
$$r_1=1,\ r_2=3,\ r_3=-2$$

$(A - rI)\bar{z} = \bar{0}$

when $r = 1$

$$\begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-z_2 + 4z_3 = 0$ (1)

$3z_1 + z_2 - z_3 = 0$ (2)

$2z_1 + z_2 - 2z_3 = 0$ (3)

If you do $2\cdot$(2) $- 3\cdot$(3), you get (1).
Hence, (1) is redundant.

$z_2 = 4z_3$

$3z_1 + 4z_3 - z_3 = 0$

$3z_1 + 3z_3 = 0$

$z_1 = -z_3$

Let $z_1 = 1$. $z_3 = -1$. $z_2 = -4$.

$$\bar{z_1} = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

when $r = 3$

$$\begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2z_1 - z_2 + 4z_3 = 0$

$3z_1 - z_2 - z_3 = 0$

$2z_1 + z_2 - 4z_3 = 0$ ← Redundant

$z_2 = -2z_1 + 4z_3$

$3z_1 - (-2z_1 + 4z_3) - z_3 = 0$

$5z_1 - 5z_3 = 0$

$z_1 = z_3$

let $z_1 = 1.$ $z_3 = 1.$ $z_2 = 2$

$$\overline{z^2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

when $r = -2$

$$\begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3z_1 - z_2 + 4z_3 = 0$ (1)

$3z_1 + 4z_2 - z_3 = 0$ (2)

$2z_1 + z_2 + z_3 = 0$ (3)

Suppose that (2) is redundant.

$z_2 = 3z_1 + 4z_3$

$2z_1 + 3z_1 + 4z_3 + z_3 = 0$

$5z_1 + 5z_3 = 0$

$z_1 = -z_3$

Let $z_1 = 1$. $z_3 = -1$. $z_2 = -1$.

$$\overline{z^3} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

The general soln is $\bar{x} = C_1 e^{t} \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} +$

$$C_2 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + C_3 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 1-r & 0 & 0 \\ 2 & 1-r & -2 \\ 3 & 2 & 1-r \end{vmatrix} = 0$$

$(1-r)\left[(1-r)^2 + 4\right] = 0$

$(1-r)(r^2 - 2r + 5) = 0$

To solve $r^2 - 2r + 5 = 0$, I'll use the quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

$r_1 = 1$, $r_2 = 1 + 2i$

$(A - rI)\bar{z} = \bar{0}$

when $r = 1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2z_1 - 2z_3 = 0 \rightarrow z_1 = z_3$

$3z_1 + 2z_2 = 0 \rightarrow \frac{-3z_1}{2} = z_2$

Let $z_1 = 2$. $z_2 = -3$, $z_3 = 2$.

$$\overline{z^1} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

When $r = 1+2i$

$$\begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2iz_1 = 0$
$2z_1 - 2iz_2 - 2z_3 = 0$
$3z_1 + 2z_2 - 2iz_3 = 0$

$z_1 = 0$
$z_2 = iz_3$
at $z_3 = 1$. $z_2 = i$

$$\overline{z^2} = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$e^{r_2 t}\, \overline{z^2}$
$= e^{(1+2i)t}\, \overline{z^2}$
$= e^{t} \cdot e^{(2t)i} \cdot \overline{z^2}$
$= e^{t} (\cos(2t) + i\sin(2t)) \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$

$$= e^t\left(\cos(2t)\begin{bmatrix}0\\0\\1\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\1\\0\end{bmatrix}\right) +$$

$$ie^t\left(\cos(2t)\begin{bmatrix}0\\1\\0\end{bmatrix} + \sin(2t)\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

The general soln is $\bar{x} = C_1e^t\begin{bmatrix}2\\-3\\2\end{bmatrix} +$

$$C_2e^t\begin{bmatrix}0\\-\sin(2t)\\\cos(2t)\end{bmatrix} + C_3e^t\begin{bmatrix}0\\\cos(2t)\\\sin(2t)\end{bmatrix}$$

E.g. Solve $\bar{x}' = \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\bar{x}$

Soln:

$$\begin{vmatrix}-r&1&1\\1&-r&1\\1&1&-r\end{vmatrix}=0$$

$$-r\left[r^2-1\right]-\left[-r-1\right]+\left[1+r\right]=0$$

$$-r^3+r+r+1+r+1=0$$

$$-r^3+3r+2=0$$

$r=-1$ is a root.

$$\begin{array}{r l}
 & -r^2 + r + 2 \\
r+1 & \overline{)\,-r^3 + 0r^2 + 3r + 2} \\
 & -(\underline{-r^3 - r^2)} \\
 & \quad r^2 + 3r \\
 & \quad -(\underline{r^2 + r)} \\
 & \qquad 2r + 2 \\
 & \qquad -(\underline{2r + 2)} \\
 & \qquad\qquad 0
\end{array}$$

$(r+1)(-r^2+r+2) = -r^3 + 3r + 2$

$(r+1)(r^2-r-2) = 0$

$(r+1)(r-2)(r+1) = 0$

$r_1 = 2,\ r_2 = r_3 = -1$

$(A - rI)\bar{z} = \bar{0}$

when $r=2$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2z_1 + z_2 + z_3 = 0$ (1)

$z_1 - 2z_2 + z_3 = 0$ (2)

$z_1 + z_2 - 2z_3 = 0$ (3)

(1) $=$ (-1) ((2) $+$ (3))

Hence, (1) is redundant.

$z_1 = 2z_2 - z_3$

$2z_2 - z_3 + z_2 - 2z_3 = 0$

$3z_2 - 3z_3 = 0$

$z_2 = z_3$

Let $z_2 = 1$. $z_1 = 1$. $z_3 = 1$

$$\overline{z_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

when $r = -1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$z_1 + z_2 + z_3 = 0$

$z_1 = -z_2 - z_3$

Let $z_2 = 0$, $z_3 = 1$. Then, $z_1 = -1$

$$\overline{z^2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $z_2 = 1$, $z_3 = 0$. Then, $z_1 = -1$

$$\overline{z^3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

The gen soln is $c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

General Theory of Linear Eqns

- An nth order linear differential eqn has the form $P_0(t)\frac{d^ny}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \dots +$ $P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t)$

- For a 1st order diff eqn, we have $P_0(t)y' + P_1(t)y = G(t)$

- For a 2nd order diff eqn, we have $P_0(t)y'' + P_1(t)y' + P_2(t)y = G(t)$

- For a 3rd order diff eqn, we have $P_0(t)y''' + P_1(t)y'' + P_2(t)y' + P_3(t)y = G(t)$

Note: We will be dealing with 3rd order linear diff eqn at most.

Note: If $P_0(t) \neq 0$, we can divide both sides of the eqn by it. Then, we get $\frac{d^ny}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \dots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$

For a 3rd order diff eqn in this form, we have $y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = g(t)$

Note: If $G(t)$ or $g(t) = 0$, then the diff eqn is homogeneous. otherwise, it's non-homogeneous.

- The existence and uniqueness thm states that given the initial conditions $Y(t_0) = y_0$, $Y'(t_0) = Y_0'$, and $Y''(t_0) = y_0''$ and assuming all functions p_k are continuous, there exists a unique soln $Y(t)$ and it is defined everywhere the eqn is defined.

- Superposition of Solns of Homogeneous Eqns: If Y_1, Y_2, Y_3 solve $Y''' + p_1(t)Y'' + p_2(t)Y' + p_3(t)y = 0$, then for any constant coefficients C_k, the linear combination $Y = C_1y_1 + C_2Y_2 + C_3Y_3$ also solves the eqn.

- The Wronksian for 3 solns is defined as

$$w[Y_1, Y_2, Y_3] = \begin{vmatrix} Y_1(t) & Y_2(t) & Y_3(t) \\ Y_1'(t) & Y_2'(t) & Y_3'(t) \\ Y_1''(t) & Y_2''(t) & Y_3''(t) \end{vmatrix}$$

Y_1, Y_2 and Y_3 is a fundamental set of solns iff $w[Y_1, Y_2, Y_3] \neq 0$.

- To get Abel's Formula, we need to differentiate w.

$$w' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}'$$

$$= \underbrace{\begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{\text{Here were differentiating the first row.}} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{\text{Here, we're differentiating the second row.}} + \underbrace{\begin{vmatrix} Y_1 & Y_2 & Y_3 \\ Y_1' & Y_2' & Y_3' \\ Y_1''' & Y_2''' & Y_3''' \end{vmatrix}}_{\text{Here, we're differentiating the 3rd row}}$$

Notice that for the first 2 determinants, there are 2 rows that are the same. Hence, the rows are linearly dependent and the det is 0. Therefore, we only care about the last det.

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} \leftarrow \text{Last determinant}$$

Recall that $y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0$.
Hence, $y''' = -p_1(t)y'' - p_2(t)y' - p_3(t)y$
Sub y''' into the determinant above.

$$-p_1 \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} - p_2 \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \end{vmatrix} - p_3 \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Equals 0 because the rows are linearly dependent

$$-p_1 \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$= -p_1 w$

$w' = -p_1 w$

$$\frac{w'}{w} = -p_1(t)$$

$$\frac{1}{w} \frac{dw}{dt} = -p_1(t)$$

$$\frac{1}{w} dw = -p_1(t)\, dt$$

$$\int \frac{1}{w} dw = \int -p_1(t)\, dt$$

$$\ln|w| + c = \int -p_1(t)\, dt$$

$$\ln|w| = c - \int p_1(t)\, dt$$

$$w = e^{c - \int p_1(t)\, dt}$$

$$= e^{c} \cdot e^{-\int p_1(t)\, dt}$$

$$= c \cdot e^{-\int p_1(t)\, dt} \leftarrow \text{Abel's Formula}$$

- Once again, there's a dichotomy. Either $w=0$ for all t or $w \neq 0$ for all t. This is because either $c'=0$ or $c' \neq 0$ and the exponent never equals to 0.

- Let Y_1, Y_2, and Y_3 be functions. Y_1, Y_2 and Y_3 are linearly independent if $C_1Y_1 + C_2Y_2 + C_3Y_3 \neq 0$ at at least 1 point unless $C_1 = C_2 = C_3 = 0$.

Another way to think about this is Y_1, Y_2, and Y_3 are linearly independent iff $w[Y_1, Y_2, Y_3] \neq 0$ at at least 1 point.

E.g. Determine if $y_1 = 2t-3$, $y_2 = t^2+1$ and $y_3 = 2t^2-t$ are linearly independent.

Soln

$$W = \begin{vmatrix} 2t-3 & t^2+1 & 2t^2-t \\ 2 & 2t & 4t-1 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= (2t-3)\left[8t-2(4t-1)\right] - (t^2+1)\left[8\right] + (2t^2-t)\left[4\right]$$
$$= (2t-3)\left[8t-8t+2\right] - 8t^2 - 8 + 8t^2 - 4t$$
$$= 4t - 6 - 8t^2 - 8 + 8t^2 - 4t$$
$$= -14$$
$$\neq 0$$

$\therefore$ y_1, y_2, y_3 are linearly independent.

E.g. Find if $y_1 = 2t-3$, $y_2 = 2t^2+1$, $y_3 = 3t^2+t$ are linearly dependent or not.

Soln:

$$W = \begin{vmatrix} 2t-3 & 2t^2+1 & 3t^2+t \\ 2 & 4t & 6t+1 \\ 0 & 4 & 6 \end{vmatrix}$$

$$= (2t-3)\left[24t-24t-4\right] - (2t^2+1)(12) + (3t^2+t)(8)$$
$$= -8t + 12 - 24t^2 - 12 + 24t^2 + 8t$$
$$= 0$$

$\therefore$ y_1, y_2 and y_3 are linearly dependent.

- When solving a third order linear diff eqn, let $y=e^{rt}$. Then, $y'=re^{rt}$, $y''=r^2e^{rt}$, and $y'''=r^3e^{rt}$.

$ay''' + by'' + cy' + dy = 0$ becomes

$ar^3e^{rt} + br^2e^{rt} + cre^{rt} + de^{rt} = 0 \iff$

$e^{rt}(ar^3 + br^2 + cr + d) = 0 \iff$

$ar^3 + br^2 + cr + d = 0$

- There are a few possible types of answers.

1. Real and distinct roots. Here, $r_1 \neq r_2$, $r_1 \neq r_3$ and $r_2 \neq r_3$. In this case, the gen soln is $C_1e^{r_1t} + C_2e^{r_2t} + C_3e^{r_3t}$.

2. Repeated roots. There are 2 possibilities:
   a) A root is repeated once. I.e. $r_1 = r_2$, but $r_1 \neq r_3$. In this case, the gen soln is $C_1e^{r_1t} + C_2te^{r_1t} + C_3e^{r_3t}$.

   b) A root is repeated twice. I.e. $r_1 = r_2 = r_3$ In this case, the gen soln is $C_1e^{r_1t} + C_2e^{r_1t}t + C_3e^{r_1t}t^2$.

3. Complex roots. Note that with complex roots, it may not be the case that all 3 roots are complex.

E.g. Find the gen soln of $y''' - y'' - y' + y = 0$.

Soln:

$r^3 - r^2 - r + 1 = 0$

$r^2(r-1) - (r-1) = 0$

$(r-1)(r^2-1) = 0$

$(r-1)(r-1)(r+1) = 0$

$r_1 = 1, \ r_2 = -1$

Hence, the gen soln is $y = C_1e^t + C_2te^t + C_3e^{-t}$.

E.g. Find the gen soln of $y''' - 3y'' + 3y' - y = 0$.

Soln:

$r^3 - 3r^2 + 3r - 1 = 0$

$(r-1)^3 = 0$

$r_1 = r_2 = r_3 = 1$

Hence, the gen soln is $Y = C_1e^t + C_2te^t + C_3t^2e^t$.

E.g. Find the gen soln of $y''' - 2y'' - y' + 2y = 0$.

Soln:

$r^3 - 2r^2 - r + 2 = 0$

$r^2(r-2) - (r-2) = 0$

$(r-2)(r^2-1) = 0$

$(r-2)(r-1)(r+1) = 0$

$r_1 = 2, \ r_2 = 1, \ r_3 = -1$

Hence, the gen soln is $y = C_1e^{2t} + C_2e^t + C_3e^{-t}$

E.g. Find the gen soln of $18y''' + 21y'' + 14y' + 4y = 0$

Soln:

$18r^3 + 21r^2 + 14r + 4 = 0$

$18r^3 + 9r^2 + 12r^2 + 6r + 8r + 4 = 0$

$9r^2(2r+1) + 6r(2r+1) + 4(2r+1) = 0$

$(2r+1)(9r^2 + 6r + 4) = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-6 \pm \sqrt{36 - 144}}{18}$$
$$= \frac{-6 \pm \sqrt{-108}}{18}$$
$$= \frac{-6 \pm 6\sqrt{3}i}{18}$$
$$= \frac{-1}{3} \pm \frac{\sqrt{3}}{3}i$$

$r_1 = -\frac{1}{2}$

$r_2 = \frac{-1}{3} + \frac{\sqrt{3}}{3}i \leftarrow \lambda = \frac{-1}{3}, \ u = \frac{1}{\sqrt{3}}$

$e^{r_1 t} = e^{-t/2}$

$e^{r_2 t} = e^{-t/3}(\cos(\frac{t}{\sqrt{3}}) + i\sin(\frac{t}{\sqrt{3}}))$

The general soln is $y = C_1 e^{-t/2} + C_2 e^{-t/3}\cos(\frac{t}{\sqrt{3}}) + C_3 e^{-t/3}\sin(\frac{t}{\sqrt{3}})$.

E.g. Find the gen soln to $y''' - y'' + y' - y = 0$

Soln:

$r^3 - r^2 + r - 1 = 0$

$r^2(r-1) + (r-1) = 0$

$(r-1)(r^2+1) = 0$

$r_1 = 1$

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$r_2 = i \leftarrow \lambda = 0, u = 1$

$e^{r_1 t} = e^t$

$e^{r_2 t} = e^{\lambda t}(\cos(ut) + i\sin(ut))$

$= \cos(t) + i\sin(t)$

Hence, the gen soln is $C_1e^t + C_2\cos(t) + C_3\sin(t)$

E.g. Find the gen soln of $y''' - y = 0$.

Soln:

$r^3 - 1 = 0$

$(r-1)(r^2 + r + 1) = 0$ Note: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$r_1 = 1$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2} \leftarrow \lambda = \frac{-1}{2}, u = \frac{\sqrt{3}}{2}$$

$e^{r_1 t} = e^{t}$

$e^{r_2 t} = e^{\lambda t}(\cos(ut) + i\sin(ut))$

$= e^{-t/2}(\cos(\frac{\sqrt{3}t}{2}) + i\sin(\frac{\sqrt{3}t}{2}))$

Hence, the general soln is $C_1e^{t} + C_2e^{-t/2}\cos(\frac{\sqrt{3}t}{2})$ $+ C_3e^{-t/2}\sin(\frac{\sqrt{3}t}{2})$.

E.g. Find the gen soln of $y''' + 5y'' + 6y' + 2y = 0$

Soln:

$r^3 + 5r^2 + 6r + 2 = 0$

$r = -1$ is a root.

$(-1)^3 + 5(-1)^2 + 6(-1) + 2$

$= -1 + 5 - 6 + 2$

$= -7 + 7$

$= 0$

$$\begin{array}{rl} & r^2 + 4r + 2 \\ r+1 \big) & r^3 + 5r^2 + 6r + 2 \\ & -(r^3 + r^2) \\ & \quad 4r^2 + 6r \\ & \quad -(4r^2 + 4r) \\ & \qquad 2r + 2 \\ & \qquad -(2r + 2) \\ & \qquad\quad 0 \end{array}$$

$(r+1)(r^2 + 4r + 2) = r^3 + 5r^2 + 6r + 2$

$= 0$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-4 \pm \sqrt{16-8}}{2}$$
$$= \frac{-4 \pm \sqrt{8}}{2}$$
$$= \frac{-4 \pm 2\sqrt{2}}{2}$$
$$= -2 \pm \sqrt{2}$$

$r_2 = -2+\sqrt{2}$

$r_3 = -2-\sqrt{2}$

The gen soln is $C_1e^{-t} + C_2e^{(-2+\sqrt{2})t} + C_3e^{(-2-\sqrt{2})t}$.

E.g. Find the gen soln to $y''' + y' = 0$

Soln:

$r^3 + r = 0$

$r(r^2+1) = 0$

$r_1 = 0$

$r^2+1=0 \rightarrow r^2=-1 \rightarrow r = \pm i$ $\lambda = 0, u = 1$

$e^{r_1t} = 1$

$e^{r_2t} = e^{\lambda t}(\cos(ut) + i\sin(ut))$

$= \cos(t) + i\sin(t)$

The gen soln is $C_1 + C_2\cos(t) + C_3\sin(t)$

E.g. Find the gen soln to $4y''' + y' + 5y = 0$.

Soln:

$4r^3 + r + 5 = 0$

$r = -1$ is a root.

$$\begin{array}{r l}
 & \quad 4r^2 - 4r + 5 \\
r+1 & \overline{)\,4r^3 + 0r^2 + r + 5} \\
 & -\underline{(4r^3 + 4r^2)} \\
 & \qquad -4r^2 + r \\
 & \quad -\underline{(-4r^2 - 4r)} \\
 & \qquad\qquad 5r + 5 \\
 & \qquad\quad -\underline{(5r + 5)} \\
 & \qquad\qquad\qquad 0
\end{array}$$

$$(r+1)(4r^2 - 4r + 5) = 4r^3 + r - 5$$
$$= 0$$

$r_1 = -1$

$e^{r_1 t} = e^{-t}$

$$\begin{aligned}
r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{4 \pm \sqrt{16 - 80}}{8} \\
&= \frac{4 \pm \sqrt{-64}}{8} \\
&= \frac{4 \pm 8i}{8} \\
&= \frac{1}{2} \pm i \leftarrow \lambda = \frac{1}{2},\ \mu = 1
\end{aligned}$$

$$e^{r_2 t} = e^{\lambda t} (\cos(ut) + i\sin(ut))$$
$$= e^{t/2} (\cos(t) + i\sin(t))$$

The gen soln is $y = C_1 e^{-t} + C_2 e^{t/2} \cos(t) + C_3 e^{t/2} \sin(t)$.

E.g. Find the gen soln to $6y''' + 5y'' + y' = 0$.

Soln:

$6r^3 + 5r^2 + r = 0$

$r(6r^2 + 5r + 1) = 0$

$r_1 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-5 \pm \sqrt{25 - 24}}{12}$$
$$= \frac{-5 \pm 1}{12}$$
$$= -\tfrac{1}{2} \text{ or } -\tfrac{1}{3}$$

$r_2 = -\frac{1}{2}$, $r_3 = -\frac{1}{3}$

The gen soln is $C_1 + C_2 e^{-t/2} + C_3 e^{-t/3}$

Series Soln Near An Ordinary Point

- Consider the diff eqn $p(x)y'' + q(x)y' + r(x)y = 0$ where p, q and r are nonconstant coefficients. We say that x_0 is an ordinary point if $p(x_0) \neq 0$ and a singular point if $p(x_0) = 0$.

E.g. Find a series soln to $y'' - xy = 0$ near $x=0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$y'' - xy = 0$ can be rewritten as

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

We want both summations to have x^n, so we tweak them

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = \underbrace{a_2(2)(1)x^0}_{\text{when } n=0} + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n$$

$$2a_2 + \sum_{n=1}^{\infty} x^n \left(a_{n+2}(n+2)(n+1) - a_{n-1}\right) = 0$$

Take $n=0$:

$2a_2 = 0$

$a_2 = 0$

Take $n \geq 1$

$a_{n+2}(n+2)(n+1) - a_{n-1} = 0$

$a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}$ ← Recurrence Equation

Let's plug some values for n.

$n=1$: $a_3 = \dfrac{a_0}{(3)(2)}$	$n=2$: $a_4 = \dfrac{a_1}{(4)(3)}$	$n=3$: $a_5 = \dfrac{a_2}{(5)(4)} = 0$
$n=4$: $a_6 = \dfrac{a_3}{(6)(5)}$ $= \dfrac{a_0}{(6)(5)(3)(2)}$	$n=5$: $a_7 = \dfrac{a_4}{(7)(6)}$ $= \dfrac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}$	$n=6$: $a_8 = \dfrac{a_5}{8 \cdot 7}$ $= 0$
$n=7$: $a_9 = \dfrac{a_6}{9 \cdot 8}$ $= \dfrac{a_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}$	$n=8$: $a_{10} = \dfrac{a_7}{10 \cdot 9}$ $= \dfrac{a_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}$	$n=9$: $a_{11} = \dfrac{a_8}{11 \cdot 10}$ $= 0$

We see a pattern:

$a_{3k+2} = 0, \ k = 0, 1, 2, \ldots$

$$a_{3k+1} = \frac{a_{3k-2}}{(3k+1)(3k)}$$

$$= \frac{a_1}{(3k+1)(3k)\ldots(4)(3)}$$

$$a_{3k} = \frac{a_{3k-3}}{(3k)(3k-1)}$$

$$= \frac{a_0}{(3k)(3k-1)\ldots(3)(2)}$$

Recall that $y = \sum_{n=0}^{\infty} a_n x^n$.

Expanding the summation, we get

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$$

$$= a_0 + a_1 x + \frac{a_0 x^3}{6} + \frac{a_1 x^4}{12} + \ldots$$

$$= a_0 \left(1 + \frac{x^3}{6} + \ldots\right) +$$

$$a_1 \left(x + \frac{x^4}{12} + \ldots\right)$$

$$y_1 = 1 + \frac{x^3}{6} + \ldots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{(2)(3)\ldots(3n-1)(3n)}$$

$$y_2 = x + \frac{x^4}{12} + \ldots$$

$$= x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(3)(4)\ldots(3n)(3n+1)}$$

Note: a_0 and a_1 are arbitrary coefficients.

$$y = a_0 y_1 + a_1 y_2$$

To see if y_1 and y_2 are a fundamental pair of solns, we will use the Wronksian.

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_1' y_2$$

Since we need to show that $w \neq 0$ at at least 1 point, we will choose an easy point, 0.

$y_1(0) = 1$

$y_2(0) = 0$

Since we already know that $y_2(0) = 0$, we don't need to calculate y_1'. It doesn't matter.

$$y_2' = 1 + \sum_{n=1}^{\infty} \frac{(3n+1)\, x^{3n}}{3 \cdot 4 \cdot \ldots \cdot 3n \cdot 3n+1}$$

$y_2'(0) = 1$

$w = 1$

$\neq 0$ $\therefore$ y_1 and y_2 are a fundamental pair of solns.

To find the convergence domain for the series

$$\sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \ldots \cdot 3n-1 \cdot 3n},$$ we do

$$\lim_{n \to \infty} \left| \frac{\dfrac{x^{3n+3}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \ldots \cdot 3(n+1)-1 \cdot 3(n+1)}}{\dfrac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \ldots \cdot 3n-1 \cdot 3n}} \right|$$

$$= \lim_{n \to \infty} \left| \left(\frac{x^{3n+3}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \ldots \cdot 3n+2 \cdot 3(n+1)} \right) \left(\frac{2 \cdot 3 \cdot 5 \cdot 6 \cdot \ldots \cdot 3n-1 \cdot 3n}{x^{3n}} \right) \right|$$

$$= \lim_{n \to \infty} \left| \frac{x^3}{(3n+2)(3n+3)} \right|$$

$$= |x^3| \lim_{n \to \infty} \left| \frac{1}{(3n+2)(3n+3)} \right|$$

$$= 0$$

∴ The radius of convergence is ∞ and the convergence domain is $-\infty < x < \infty$.

To find the convergence domain for

$$\sum_{n=1}^{\infty} \frac{x^{3n+1}}{3\cdot 4\cdot\ldots\cdot 3n\cdot 3n+1},$$ we do

$$\lim_{n\to\infty} \frac{\left|\dfrac{x^{3(n+1)+1}}{3\cdot 4\cdot\ldots\cdot 3(n+1)\cdot 3(n+1)+1}\right|}{\left|\dfrac{x^{3n+1}}{3\cdot 4\cdot\ldots\cdot 3n\cdot 3n+1}\right|}$$

$$= \lim_{n\to\infty} \left|\frac{x^{3n+4}}{3\cdot 4\cdot\ldots\cdot 3n+3\cdot 3n+4}\right| \left|\frac{3\cdot 4\cdot\ldots\cdot 3n\cdot 3n+1}{x^{3n+1}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^3}{(3n+3)(3n+4)}\right|$$

$$= |x^3| \lim_{n\to\infty} \left|\frac{1}{(3n+3)(3n+4)}\right|$$

$$= 0$$

$\therefore$ The radius of convergence is ∞ and the convergence domain is $-\infty < x < \infty$.

E.g. Determine a series soln for $y''+y=0$ at $x_0=0$.

Soln:

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=1}^{\infty} a_n(n)x^{n-1}, \quad y''=\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2}$$

We can rewrite $y''+y=0$ as

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

We want both summations to have x^n.

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (a_{n+2}(n+2)(n+1)+a_n)x^n = 0$$

when $n \geq 0$:

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)} \leftarrow \text{Recurrence Eqn}$$

Let's plug some values for n.

$n=0$: $a_2 = \frac{-a_0}{2 \cdot 1}$	$n=1$: $a_3 = \frac{-a_1}{3 \cdot 2}$
$n=2$: $a_4 = \frac{-a_2}{4 \cdot 3}$ $= \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}$	$n=3$: $a_5 = \frac{-a_3}{5 \cdot 4}$ $= \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}$
$n=4$: $a_6 = \frac{-a_4}{6 \cdot 5}$ $= \frac{-a_0}{6!}$	$n=5$: $a_7 = \frac{-a_5}{7 \cdot 6}$ $= \frac{-a_1}{7!}$

We see a pattern.

$$a_{2k} = \frac{(-1)^k \cdot a_0}{(2k)!}, \quad k=1,2,\ldots$$

$$a_{2k+1} = \frac{(-1)^k \cdot a_1}{(2k+1)!}, \quad k=1,2,\ldots$$

Recall that $y = \sum_{n=0}^{\infty} a_n x^n$.

Expanding the summation, we get

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 + a_1 x - \frac{a_0 x^2}{2!} - \frac{a_1 x^3}{3!} + \ldots$$

$$= a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right) +$$

$$a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)$$

$$= a_0 \underbrace{\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right)}_{y_1} + a_1 \underbrace{\left(x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)}_{y_2}$$

E.g. Determine a series soln for $y'' - y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

We can rewrite $y'' - y = 0$ as

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0$$

We want both summations to have x^n.

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} x^n (a_{n+2}(n+2)(n+1) - a_n) = 0$$

For $n \geq 0$:

$a_{n+2}(n+2)(n+1) - a_n = 0$

$a_{n+2} = \frac{a_n}{(n+2)(n+1)}$

Let's plug some values for n.

$n=0$: $a_2 = \frac{a_0}{2 \cdot 1}$	$n=1$: $a_3 = \frac{a_1}{3 \cdot 2}$
$n=2$: $a_4 = \frac{a_2}{4 \cdot 3}$ $= \frac{a_0}{4!}$	$n=3$: $a_5 = \frac{a_3}{5 \cdot 4}$ $= \frac{a_1}{5!}$
$n=4$: $a_6 = \frac{a_4}{6 \cdot 5}$ $= \frac{a_0}{6!}$	$n=5$: $a_7 = \frac{a_5}{7 \cdot 6}$ $= \frac{a_1}{7!}$

We see a pattern.

$$a_{2k} = \frac{a_0}{(2n)!}, \quad k=1,2,\ldots$$

$$a_{2k+1} = \frac{a_1}{(2k+1)!}, \quad k=1,2,\ldots$$

Recall $y = \sum_{n=0}^{\infty} a_n x^n$.

Expanding the summation, we get

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$= a_0 + a_1 x + \frac{a_0 x^2}{2!} + \frac{a_1 x^3}{3!} + \ldots$$

$$= a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) +$$

$$a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= \underbrace{a_0\left(\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right)}_{y_1} + \underbrace{a_1\left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\right)}_{y_2}$$

E.g. Find a series soln for $y'' + 3y' = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} a_n (n) x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

We can rewrite $y'' + 3y' = 0$ as

$$\sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + 3 \sum_{n=1}^{\infty} a_n (n) x^{n-1} = 0$$

We want both summations to have x^n.

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n + 3 \sum_{n=0}^{\infty} a_{n+1} (n+1) x^n = 0$$

$$\sum_{n=0}^{\infty} (a_{n+2} (n+2)(n+1) + 3 a_{n+1} (n+1)) x^n = 0$$

For $n \geq 0$

$$a_{n+2} (n+2)(n+1) + 3a_{n+1} (n+1) = 0$$

$$a_{n+2} = \frac{-3a_{n+1}}{n+2}$$

Now, we'll plug some values for n.

n=0:

$$a_2 = \frac{-3a_1}{2}$$

n=1:

$$a_3 = \frac{9a_1}{3 \cdot 2}$$

n=2:

$$a_4 = \frac{-3a_3}{4}$$

$$= \frac{-27a_1}{4!}$$

n=3:

$$a_5 = \frac{-3a_4}{5}$$

$$= \frac{81a_1}{5!}$$

We see a pattern.

$$a_k = \frac{(-3)^{k-1} \cdot a_1}{k!}, \quad k = 2, 3, \ldots$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= 1 + a_1 x - \frac{3a_1 x^2}{2!} + \frac{9a_1 x^3}{3!} - \dots$$

$$= 1 + a_1 \left(x - \frac{3x^2}{2!} + \frac{9x^3}{3!} - \dots \right)$$

$$y_1 = 1$$

$$y_2 = x - \frac{3x^2}{2!} + \frac{9x^3}{3!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-3)^n x^{(n+1)}}{(n+1)!}$$

Series Soln Near A Singular Point

- Consider $P(t)y'' + Q(t)y' + R(t)y = 0$.

t_0 is a singular point if $P(t_0) = 0$.

- When we want to find a series soln and we have a singular point, we let

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r}$$

- By convention, $a_0 = 1$.

E.g. Find a series soln for $2x^2y'' - xy' + (1+x)y = 0$ about $x_0 = 0$.

Soln:

x_0 is a singular point.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r}, \quad y'' = \sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $2x^2y'' - xy' + (1+x)y = 0$ as

$$\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} +$$

$$\sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

We want all summations to have x^{n+r}.

$$\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} +$$

$$\sum_{n=1}^{\infty} a_{n-1}x^{n+r} = 0$$

Take $n=0$:

$2a_0(r)(r-1) - a_0(r) + a_0 = 0$

$a_0(2r(r-1) - r + 1) = 0$

Recall: By convention, $a_0 = 1$.

$2r^2 - 2r - r + 1 = 0$ Indicial Eqn

$2r^2 - 3r + 1 = 0$

The soln to the indicial eqn is called the index or singularity exponent.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{9 - 8}}{4}$$

$$= \frac{3 \pm 1}{4}$$

$r_1 = 1$, $r_2 = \frac{1}{2}$ Note: By convention, if we have 2 real roots, we let r_1 be the bigger one and r_2 be the smaller one.

Going back to the eqn, for $n \geq 1$, we have

$$2a_n(n+r)(n+r-1) - a_n(n+r) + a_n + a_{n-1} = 0$$

$$a_n(2(n+r)(n+r-1) - (n+r) + 1) = -a_{n-1}$$

Let $a = n+r$

Then, we have $2a(a-1) - a + 1$

$$\longleftrightarrow 2a^2 - 2a - a + 1$$

$$\longleftrightarrow 2a^2 - 3a + 1$$

$$\longleftrightarrow (a-1) \qquad (2a-1)$$

$$\longleftrightarrow (n+r-1)(2(n+r)-1)$$

$$a_n = \frac{-a_{n-1}}{(n+r-1)(2(n+r)-1)}$$

Note: For singular point, we don't need to find the summations. We just need the first 2-3 values in the summation.

For $r=1$

Take $n=1$:	Take $n=2$:	Take $n=3$:
$a_1 = \frac{-a_0}{(1)(3)}$ $= \frac{-1}{(1)(3)}$	$a_2 = \frac{-a_1}{(2)(5)}$ $= \frac{1}{(1)(2)(3)(5)}$	$a_3 = \frac{-a_2}{(3)(7)}$ $= \frac{-1}{1 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots \leftarrow r=1$$

$$= x - \frac{x^2}{3} + \frac{x^3}{30} - \frac{x^4}{630} + \ldots$$

For $r = \frac{1}{2}$:

Take $n=1$:	Take $n=2$:	Take $n=3$:
$a_1 = \frac{-a_0}{(\frac{1}{2})(2)}$ $= -1$	$a_2 = \frac{-a_1}{(\frac{3}{2})(4)}$ $= \frac{1}{6}$	$a_3 = \frac{-a_2}{(\frac{5}{2})(6)}$ $= \frac{-1}{90}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_2 = a_0 x^{1/2} + a_1 x^{3/2} + a_2 x^{5/2} + a_3 x^{7/2} + \ldots \leftarrow r = \frac{1}{2}$$
$$= x^{1/2} - x^{3/2} + \frac{x^{5/2}}{6} - \frac{x^{7/2}}{90} + \ldots$$

$$y = c_1 y_1 + c_2 y_2$$

– There are 3 cases we have to deal with:

1. $r_1, r_2 \in R$ and $r_1 \neq r_2$ and $r_1 - r_2$ is not an integer
2. $r_1, r_2 \in R$ and $r_1 = r_2$
3. $r_1, r_2 \in R$ and $r_1 \neq r_2$ and $r_1 - r_2$ is an integer

Case 1:

The first case occurs when r_1 and r_2 are real numbers and they don't equal each other and their difference is not an integer.

E.g. Find a series soln for $2xy'' + y' + xy = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $2xy'' + y' + xy = 0$ as

$$\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

We want all summations to have x^{n+r-1}.

$$\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

Take $n=0$:

$2a_0(r)(r-1) + a_0(r) = 0$

$a_0(2r(r-1) + r) = 0$

$2r^2 - 2r + r = 0$

$2r^2 - r = 0$

$r(2r-1) = 0$

$r_1 = \frac{1}{2}, \quad r_2 = 0$

In this case, $a_1 = 0$.

Take $n \geq 2$:

$2a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} = 0$

$a_n\left(2(n+r)(n+r-1) + (n+r)\right) = -a_{n-2}$

Let $a = n+r$

$2a(a-1) + a$

$= 2a^2 - 2a + a$

$= 2a^2 - a$

$= a(2a-1)$

$= (n+r)(2(n+r)-1)$

$$a_n = \frac{-a_{n-2}}{(n+r)(2(n+r)-1)} \leftarrow \text{Recurrence Eqn}$$

For $r = \frac{1}{2}$:

Take $n = 2$:

$$a_2 = \frac{-a_0}{(\frac{5}{2})(4)} = \frac{-1}{10}$$

Take $n = 4$:

$$a_4 = \frac{-a_2}{(\frac{9}{2})(8)} = \frac{1}{360}$$

Take $n = 6$:

$$a_6 = \frac{-a_4}{(\frac{13}{2})(12)} = \frac{-1}{28{,}080}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{1/2} + a_2 x^{5/2} + a_4 x^{9/2} + a_6 x^{13/2} + \ldots \leftarrow r = \tfrac{1}{2}$$

$$= x^{1/2} - \frac{x^{5/2}}{10} + \frac{x^{9/2}}{360} - \frac{x^{13/2}}{28{,}080} + \ldots$$

For $r = 0$

Take $n=2$:	Take $n=4$:	Take $n=6$:
$a_2 = \frac{-a_0}{(2)(3)}$	$a_4 = \frac{-a_2}{4(7)}$	$a_6 = \frac{-a_4}{6(11)}$
$= \frac{-1}{6}$	$= \frac{1}{168}$	$= \frac{-1}{11{,}080}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_2 = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \dots \leftarrow r=0$$

$$= 1 - \frac{x^2}{6} + \frac{x^4}{168} - \frac{x^6}{11{,}080} + \dots$$

$$y = c_1 y_1 + c_2 y_2$$

E.g. Find a series soln to $4xy'' + 2y' + y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $4xy'' + 2y' + y = 0$ as

$$\sum_{n=0}^{\infty} 4a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n (n+r) x^{n+r-1} +$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

We want all summations to have x^{n+r-1}.

$$\sum_{n=0}^{\infty} 4a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n(n+r)x^{n+r-1} +$$

$$\sum_{n=1}^{\infty} a_{n-1}x^{n+r-1} = 0$$

Take $n=0$:

$4a_0(r)(r-1) + 2a_0(r) = 0$

$a_0(4r(r-1) + 2r) = 0$

$4r^2 - 4r + 2r = 0$

$4r^2 - 2r = 0$

$r(2r-1) = 0$

$r_1 = \frac{1}{2}, \quad r_2 = 0$

Take $n \geq 1$

$4a_n(n+r)(n+r-1) + 2a_n(n+r) + a_{n-1} = 0$

$a_n(4(n+r)(n+r-1) + 2(n+r)) = -a_{n-1}$

Let $a = n+r$

$4a(a-1) + 2a$

$= 4a^2 - 4a + 2a$

$= 4a^2 - 2a$

$= (2a)(2a-1)$

$= (2(n+r))(2(n+r)-1)$

$$a_n = \frac{-a_{n-1}}{(2(n+r))(2(n+r)-1)}$$

For $r=\frac{1}{2}$:

Take $n=1$:	Take $n=2$:	Take $n=3$:
$a_1 = \frac{-a_0}{(3)(2)}$ $= \frac{-1}{6}$	$a_2 = \frac{-a_1}{(5)(4)}$ $= \frac{1}{120}$	$a_3 = \frac{-a_2}{(7)(6)}$ $= \frac{-1}{5040}$

For $r=0$:

Take $n=1$:	Take $n=2$:	Take $n=3$:
$a_1 = \frac{-a_0}{(2)(1)}$ $= \frac{-1}{2}$	$a_2 = \frac{-a_1}{4 \cdot 3}$ $= \frac{1}{24}$	$a_3 = \frac{-a_2}{6 \cdot 5}$ $= \frac{-1}{720}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{1/2} + a_1 x^{3/2} + a_2 x^{5/2} + a_3 x^{7/2} + \ldots \leftarrow r = \frac{1}{2}$$
$$= x^{1/2} - \frac{x^{3/2}}{6} + \frac{x^{5/2}}{120} - \frac{x^{7/2}}{5040} + \ldots$$

$$y_2 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \leftarrow r=0$$
$$= 1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \ldots$$

$$y = c_1 y_1 + c_2 y_2$$

Case 2:

The second case occurs when we have repeated roots. We will have to use the **Frobenius Method** to find the second soln.

E.g. Find a series soln for $xy'' + y' - y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $xy'' + y' - y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} -$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

We want all summations to have x^{n+r-1}

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} -$$

$$\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

Take $n=0$:

$a_0(r)(r-1) + a_0(r) = 0$

$a_0(r(r-1)+r) = 0$

$r^2 - r + r = 0$

$r^2 = 0$

$r_1 = r_2 = 0$ ← Repeated Roots

Take $n \geq 1$:

$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1} = 0$

$a_n((n+r)(n+r-1) + (n+r)) = a_{n-1}$

Let $a = n+r$

$a(a-1)+a$

$= a^2 - a + a$

$= a^2 = (n+r)^2$

$$a_n = \frac{a_{n-1}}{(n+r)^2}$$

For $r=0$:

$n=1$:	$n=2$:	$n=3$:
$a_1 = \frac{a_0}{1^2}$	$a_2 = \frac{a_1}{2^2}$	$a_3 = \frac{a_2}{3^2}$
$= 1$	$= \frac{1}{4}$	$= \frac{1}{36}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$y_1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$ ← $r_1 = 0$

$= 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \ldots$

Now, we'll use the **Frobenius Method** to find Y_2.

$$y_1 = \sum_{n=0}^{\infty} a_n(r) x^{n+r} \quad \text{where } a_n(r) = \frac{a_{n-1}(r)}{(n+r)^2}$$

$$y_2 = \log(x) y_1 + \sum_{n=0}^{\infty} a_n'(r) x^{n+r} \Big|_{r=r_1}$$

$$\begin{aligned} a_1' &= \partial r(a_1) \\ &= \partial r\left(\frac{a_0}{(n+r)^2}\right) \\ &= \partial r\left(\frac{1}{(1+r)^2}\right) \\ &= \frac{-2}{(1+r)^3} \end{aligned}$$

$$\begin{aligned} a_1' x^{1+r} \Big|_{r=r_1} &= \frac{-2}{(1+0)^3} x^1 \\ &= -2x \end{aligned}$$

$$\begin{aligned} a_2' &= \partial r(a_2) \\ &= \partial r\left(\frac{a_1}{(2+r)^2}\right) \\ &= \partial r\left(\frac{1}{(2+r)^2(1+r)^2}\right) && \text{Because } a_1 = \frac{1}{(1+r)^2} \\ &= \partial r\left(\frac{1}{(r^2+3r+2)^2}\right) && \text{Because } a^2 \cdot b^2 = (ab)^2 \end{aligned}$$

$$= \frac{-2(r^2+3r+2)(2r+3)}{(r^2+3r+2)^4}$$

$$= \frac{-2(2r+3)}{(r^2+3r+2)^3}$$

$$a_2' x^{2+r} |_{r=r_1} = \frac{-2(0+3)}{(0+0+2)^3} x^2$$

$$= \frac{-6}{8} x^2$$

$$= \frac{-3}{4} x^2$$

$$y_2 = \log(x) y_1 - 2x - \frac{3x^2}{4} + \ldots$$

E.g. Find a series soln to $xy'' + y' + xy = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $xy'' + y' + xy = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} +$$

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

For $n=0$:

$a_0(r)(r-1)+a_0(r)=0$

$a_0(r^2-r+r)=0$

$r^2=0$

$r_1=r_2=0$ ← Repeated Roots

For $n\geq 2$:

$a_n(n+r)(n+r-1)+a_n(n+r)+a_{n-2}=0$

$a_n((n+r)(n+r-1)+(n+r))=-a_{n-2}$

Let $a=n+r$

$a(a-1)+a$

$=a^2$

$=(n+r)^2$

$$a_n=\frac{-a_{n-2}}{(n+r)^2}$$

Note: $a_1=0$

For $r=0$:

$n=2$:

$$a_2=\frac{-a_0}{4}$$
$$=-\frac{1}{4}$$

$n=4$:

$$a_4=\frac{-a_2}{16}$$
$$=\frac{1}{2^2\cdot 4^2}$$

$n=6$:

$$a_6=\frac{-a_4}{6^2}$$
$$=\frac{-1}{2^2\cdot 4^2\cdot 6^2}$$

$$y_1=1-\frac{x^2}{4}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\ldots$$

$$y_2 = \log(x)\, y_1 + \sum_{n=0}^{\infty} a_n'(r)\, x^{n+r} \Big|_{r=r_1}$$

$$\begin{aligned} a_2' &= \partial r(a_2) \\ &= \partial r\left(\frac{-a_0}{(2+r)^2}\right) \\ &= \partial r\left(\frac{-1}{(2+r)^2}\right) \\ &= \frac{2(2+r)}{(2+r)^4} \\ &= \frac{2}{(2+r)^3} \end{aligned}$$

$$\begin{aligned} a_2' x^{2+r} \Big|_{r=r_1} &= \frac{2}{(2+0)^3} x^{2+0} \\ &= \frac{2}{8} x^2 \\ &= \frac{x^2}{4} \end{aligned}$$

$$\begin{aligned} a_4' &= \partial r(a_4) \\ &= \partial r\left(\frac{-a_2}{(4+r)^2}\right) \\ &= \partial r\left(\frac{1}{(4+r)^2(2+r)^2}\right) \\ &= \partial r\left(\frac{1}{(r^2+6r+8)^2}\right) \\ &= \frac{-2(r^2+6r+8)(2r+6)}{(r^2+6r+8)^4} \\ &= \frac{-2(2r+6)}{(r^2+6r+8)^3} \end{aligned}$$

$$a_4' x^{4+r} \big|_{r=r_1} = \frac{-2(6)}{8^3} x^4$$
$$= \frac{-12}{512} x^4$$
$$= \frac{-3x^4}{128}$$

$$y_2 = \log(x) y_1 + \frac{x^2}{4} - \frac{3x^4}{128} + \dots$$

Case 3:

The third and final case occurs when $r_1 - r_2$ is an integer greater than 0. Here, we will use the Frobenius Method again. The **Frobenius Theorem** states that there is always a linearly independent soln

$$y_2 = a \log(x) y_1 + x^{r_2} \left(1 + \sum_{n=1}^{\infty} c_n(r_2) x^n\right) \text{ where}$$

$$a = \lim_{r \to r_2} (r - r_2) a_N(r), \ N = r_1 - r_2$$

$$c_n = \left[(r - r_2) a_n(r)\right]' \big|_{r=r_2}$$

Note: a could be 0.

E.g. Find a series soln to $xy'' - y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $xy'' - y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

Take $n=0$:

$a_0 (r)(r-1) = 0$

$r_1 = 1, \ r_2 = 0$

Take $n \geq 1$:

$a_n (n+r)(n+r-1) - a_{n-1} = 0$

$$a_n = \frac{a_{n-1}}{(n+r)(n+r-1)}$$

For $r=1$:

Take $n=1$:	Take $n=2$:	Take $n=3$:
$a_1 = \frac{a_0}{(2)(1)}$	$a_2 = \frac{a_1}{(3)(2)}$	$a_3 = \frac{a_2}{(4)(3)}$
$= \frac{1}{2}$	$= \frac{1}{12}$	$= \frac{1}{144}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots$$
$$= x + \frac{x^2}{2} + \frac{x^3}{12} + \frac{x^4}{144} + \ldots$$

$$y_2 = a \log(x) y_1 + x^0 \left(1 + \sum_{n=1}^{\infty} C_n(r_2) x^n\right)$$

$$a = \lim_{r \to 0} (r-0) a_1(r)$$
$$= \lim_{r \to 0} r \left(\frac{a_0}{(1+r)(r)}\right)$$
$$= \lim_{r \to 0} \left(\frac{1}{1+r}\right)$$
$$= 1$$

$$C_n = \left[(r-r_2) a_n(r)\right]' \Big|_{r=r_2}$$

$$C_1 = \left[(r-0) a_1(r)\right]' \Big|_{r=0}$$
$$= \left(r \left(\frac{1}{(1+r)(r)}\right)\right)' \Bigg|_{r=0}$$
$$= \left(\frac{1}{1+r}\right)' \Bigg|_{r=0}$$
$$= \frac{-1}{(1+r)^2} \Bigg|_{r=0}$$
$$= -1$$

$$C_2 = \left[(r-r_0)\, a_2(r) \right]' \Big|_{r=0}$$

$$= \left(r \left(\frac{a_1}{(2+r)(r+1)} \right) \right)' \Big|_{r=0}$$

$$= \left(r \left(\frac{1}{(2+r)(1+r)^2 (r)} \right) \right)' \Big|_{r=0}$$

$$= \left(\frac{1}{(r+1)^2 (r+2)} \right)' \Big|_{r=0}$$

$$= \left(\frac{1}{(r^2+2r+1)(r+2)} \right)' \Big|_{r=0}$$

$$= \left(\frac{1}{r^3+2r^2+2r^2+4r+r+2} \right)' \Big|_{r=0}$$

$$= \left(\frac{1}{r^3+4r^2+5r+2} \right)' \Big|_{r=0}$$

$$= \frac{-(3r^2+8r+5)}{(r^3+4r^2+5r+2)^2} \Big|_{r=0}$$

$$= \frac{-5}{4}$$

$$y_2 = \log(x)\, y_1 - x - \frac{5x^2}{4} - \ldots$$

More Examples

E.g. Find a series soln for $2xy'' + y' + xy = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $2xy'' + y' + xy = 0$ as

$$\sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} +$$

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

When $n=0$:

$a_0(2r(r-1)+r)=0$

$2r^2 - 2r + r = 0$

$2r^2 - r = 0$

$r(2r-1) = 0$

$r_1 = \frac{1}{2}, \ r_2 = 0$

when $n \geq 2$:

$$2a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} = 0$$

$$a_n\left(2(n+r)(n+r-1) + (n+r)\right) = -a_{n-2}$$

Let $a = n+r$

$$2a(a-1)+a$$

$$2a^2 - 2a + a$$

$$2a^2 - a$$

$$a(2a-1)$$

$$(n+r)(2(n+r)-1)$$

$$a_n = \frac{-a_{n-2}}{(n+r)(2(n+r)-1)}$$

$$a_1 = 0$$

when $r = \frac{1}{2}$:

$n=2$:

$$a_2 = \frac{-a_0}{(\frac{5}{2})(4)} = \frac{-1}{10}$$

$n=4$:

$$a_4 = \frac{-a_2}{(\frac{9}{2})(8)} = \frac{1}{360}$$

$n=6$:

$$a_6 = \frac{-a_4}{(\frac{13}{2})(12)} = \frac{1}{28{,}080}$$

when $r=0$:

$n=2$:

$$a_2 = \frac{-a_0}{2 \cdot 3} = \frac{-1}{6}$$

$n=4$:

$$a_4 = \frac{-a_2}{4 \cdot 7} = \frac{1}{168}$$

$n=6$:

$$a_6 = \frac{-a_4}{6 \cdot 11} = \frac{-1}{11{,}080}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{1/2} + a_2 x^{5/2} + a_4 x^{9/2} + a_6 x^{13/2} + \ldots \quad \leftarrow r = \tfrac{1}{2}$$

$$= x^{1/2} - \frac{x^{5/2}}{10} + \frac{x^{9/2}}{360} - \frac{x^{13/2}}{28{,}080} + \ldots$$

$$y_2 = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \dots \leftarrow r=0$$

$$= 1 - \frac{x^2}{6} + \frac{x^4}{168} - \frac{x^6}{11{,}080} + \dots$$

E.g. Find a series soln to $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} +$$

$$\underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+2}} - \sum_{n=0}^{\infty} \frac{1}{9} a_n x^{n+r} = 0$$

$\downarrow$ We want all summations to have x^{n+r}.

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Take n=0:

$a_0(r)(r-1) + a_0(r) - \frac{1}{9}a_0 = 0$

$a_0\left(r(r-1) + r - \frac{1}{9}\right) = 0$

$r^2 - r + r - \frac{1}{9} = 0$

$r^2 = \frac{1}{9}$

$r_1 = \frac{1}{3}, \ r_2 = -\frac{1}{3}$

Take n≥2:

$a_n(n+r)(n+r-1) + a_n(n+r) - \frac{1}{9}a_n + a_{n-2} = 0$

$a_n\left((n+r)(n+r-1) + (n+r) - \frac{1}{9}\right) = -a_{n-2}$

Let $a = n+r$

$a(a-1) + a - \frac{1}{9}$

$a^2 - a + a - \frac{1}{9}$

$a^2 - \frac{1}{9}$

$(a - \frac{1}{3})(a + \frac{1}{3})$

$(n+r-\frac{1}{3})(n+r+\frac{1}{3})$

$$a_n = \frac{-a_{n-2}}{(n+r-\frac{1}{3})(n+r+\frac{1}{3})}$$

$r = \frac{1}{3}$

n=2:

$$a_2 = \frac{-a_0}{(2)(\frac{8}{3})} = \frac{-1}{16/3} = -3/16$$

n=4:

$$a_4 = \frac{-a_2}{(4)(\frac{14}{3})}$$

$$= \left(\frac{3}{16}\right) / \left(\frac{56}{3}\right)$$

$$= \frac{9}{896}$$

$r = -\frac{1}{3}$

$n=2$:

$$a_2 = \frac{-a_0}{(2)(2-\frac{2}{3})} = \frac{-1}{(2)(2-\frac{2}{3})}$$

$n=4$:

$$a_4 = \frac{-a_2}{(4)(4-\frac{2}{3})} = \frac{1}{(2)(4)(2-\frac{2}{3})(4-\frac{2}{3})}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{1/3} + a_2 x^{8/3} + a_4 x^{13/3} + \dots \leftarrow r = \frac{1}{3}$$
$$= x^{1/3} - \frac{3x^{8/3}}{16} + \frac{9x^{13/3}}{896} - \dots$$

$$y_2 = a_0 x^{-1/3} + a_2 x^{5/3} + a_4 x^{11/3} \leftarrow r = -\frac{1}{3}$$
$$= x^{-1/3} - \frac{x^{5/3}}{(2)(2-\frac{2}{3})} + \frac{x^{11/3}}{8(2-\frac{2}{3})(4-\frac{2}{3})} - \dots$$

E.g. Find a series soln to $x^2y'' + xy' + (x-2)y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $x^2y'' + xy' + (x-2)y = 0$ as

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=1}^{\infty} a_{n-1}x^{n+r}$$

$$- \sum_{n=0}^{\infty} 2a_nx^{n+r} = 0$$

Take $n=0$:

$a_0(r)(r-1) + a_0(r) - 2a_0 = 0$

$a_0(r(r-1) + r - 2) = 0$

$r^2 - r + r - 2 = 0$

$r^2 = 2$

$r_1 = \sqrt{2}, \quad r_2 = -\sqrt{2}$

Take $n \geq 1$:

$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-1} - 2a_n = 0$

$a_n((n+r)(n+r-1) + (n+r) - 2) = -a_{n-1}$

Let $a = n+r$

$a(a-1) + a - 2$

$a^2 - a + a - 2$

$a^2 - 2$

$(a - \sqrt{2})(a + \sqrt{2})$

$(n+r-\sqrt{2})(n+r+\sqrt{2})$

$$a_n = \frac{-a_{n-1}}{(n+r-\sqrt{2})(n+r+\sqrt{2})}$$

For $r=\sqrt{2}$

$n=1$:

$$a_1 = \frac{-a_0}{(1)(1+2\sqrt{2})} = \frac{-1}{2\sqrt{2}+1}$$

$n=2$:

$$a_2 = \frac{-a_1}{(2)(2+2\sqrt{2})} = \frac{1}{2 \quad (2+2\sqrt{2})(1+2\sqrt{2})}$$

For $r=-\sqrt{2}$

$n=1$:

$$a_1 = \frac{-a_0}{(1-2\sqrt{2})(1)} = \frac{-1}{1-2\sqrt{2}}$$

$n=2$:

$$a_2 = \frac{-a_1}{(2-2\sqrt{2})(2)} = \frac{1}{(2)(1-2\sqrt{2})(2-2\sqrt{2})}$$

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{\sqrt{2}} + a_1 x^{1+\sqrt{2}} + a_2 x^{2+\sqrt{2}} + \dots \leftarrow r=\sqrt{2}$$
$$= x^{\sqrt{2}} - \frac{x^{1+\sqrt{2}}}{1+2\sqrt{2}} + \frac{x^{2+\sqrt{2}}}{2(1+2\sqrt{2})(2+2\sqrt{2})} - \dots$$

$$y_2 = a_0 x^{-\sqrt{2}} + a_1 x^{1-\sqrt{2}} + a_2 x^{2-\sqrt{2}} + \dots \leftarrow r=-\sqrt{2}$$
$$= x^{-\sqrt{2}} - \frac{x^{1-\sqrt{2}}}{1-2\sqrt{2}} + \frac{x^{2-\sqrt{2}}}{(2)(1-2\sqrt{2})(2-2\sqrt{2})}$$

E.g. Find a series soln to $xy'' + y' - y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1},$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $xy'' + y' - y = 0$ as

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - \sum_{n=1}^{\infty} a_{n-1}x^{n+r-1} = 0$$

Take $n=0$:

$a_0(r)(r-1) + a_0(r) = 0$

$a_0(r^2 - r + r) = 0$

$r^2 = 0$

$r_1 = r_2 = 0$

Take $n \geq 1$:

$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1} = 0$

$a_n((n+r)(n+r-1) + (n+r)) = a_{n-1}$

Let $a = n+r$

$a(a-1) + a$

$a^2 - a + a$

a^2

$(n+r)^2$

$$a_n = \frac{-a_{n-1}}{(n+r)^2}$$

For $r=0$:

$n=1$:	$n=2$:	$n=3$:
$a_1 = \frac{a_0}{1^2}$	$a_2 = \frac{a_1}{2^2}$	$a_3 = \frac{a_2}{3^2}$
$= 1$	$= \frac{1}{4}$	$= \frac{1}{36}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$
$$= 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \ldots$$

Since $r_1 = r_2$, we need to use Frobenius Method to find Y_2.

$$Y_2 = \log(x)\, y_1 + \sum_{n=0}^{\infty} a_n'(r) x^{n+r} \Big|_{r=r_1}$$

$$a_1' = \partial r\,(a_1)$$
$$= \partial r \left(\frac{a_0}{(1+r)^2} \right)$$
$$= \partial r \left(\frac{1}{(1+r)^2} \right)$$
$$= \frac{-2(1+r)}{(1+r)^4}$$
$$= \frac{-2}{(1+r)^3}$$

$$a_1' x^{1+r} \Big|_{r=r_1} = -2x$$

$$\begin{aligned} a_2' &= \partial r(a_2) \\ &= \partial r\left(\frac{a_1}{(2+r)^2}\right) \\ &= \partial r\left(\frac{a_0}{(1+r)^2(2+r)^2}\right) \\ &= \partial r\left(\frac{1}{(r^2+3r+2)^2}\right) \end{aligned}$$

Recall: $a^2 \cdot b^2 = (a \cdot b)^2$

$$\begin{aligned} &= \frac{2(r^2+3r+2)(2r+3)}{(r^2+3r+2)^4} \\ &= \frac{2(2r+3)}{(r^2+3r+2)^3} \end{aligned}$$

$$\begin{aligned} a_2' x^{2+r} \big|_{r=0} &= \frac{2(3)}{8} x^2 \\ &= \frac{3}{4} x^2 \end{aligned}$$

$$y_2 = \log(x)\, y_1 - 2x + \frac{3}{4}x^2 - \cdots$$

E.g. Find a series soln to $xy'' + y = 0$ about $x_0 = 0$.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $xy'' + y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0$$

Take $n=0$:

$a_0(r)(r-1)=0$

$r_1=1,\ r_2=0$

Take $n\geq 1$:

$a_n(n+r)(n+r-1)+a_{n-1}=0$

$$a_n=\frac{-a_{n-1}}{(n+r)(n+r-1)}$$

$r=1$:

$n=1$:	$n=2$:	$n=3$:
$a_1=\dfrac{-a_0}{(2)(1)}$	$a_2=\dfrac{-a_1}{(3)(2)}$	$a_3=\dfrac{-a_2}{(4)(3)}$
$=\dfrac{-1}{2}$	$=\dfrac{1}{12}$	$=\dfrac{-1}{144}$

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$

$$y_1=a_0x+a_1x^2+a_2x^3+a_3x^4+\ldots$$
$$=x-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{144}+\ldots$$

$$y_2=a\log(x)y_1+x^{r_2}\left(1+\sum_{n=1}^{\infty}c_n(r)x^n\right)$$

$$a=\lim_{r\to r_2}(r-r_2)a_N(r)$$

$$=\lim_{r\to 0}(r-0)a_1(r)$$

$$=\lim_{r\to 0}r\left(\frac{-a_0}{(1+r)(r)}\right)$$

$$=\lim_{r\to 0}\left(\frac{-1}{1+r}\right)=-1$$

$$c_n = \left[(r-r_2)a_n(r)\right]'\Big|_{r=r_2}$$

$$c_1 = \left[(r-0)\,a_1(r)\right]'\Big|_{r=r_2}$$

$$= \left(r\left(\frac{-a_0}{(1+r)(r)}\right)\right)'\Big|_{r=r_2}$$

$$= \left(\frac{-1}{(1+r)}\right)'\Big|_{r=r_2}$$

$$= \frac{1}{(1+r)^2}\Big|_{r=r_2}$$

$$= 1$$

$$c_2 = \left[(r-r_2)a_2(r)\right]'\Big|_{r=r_2}$$

$$= \left[r a_2(r)\right]'\Big|_{r=r_2}$$

$$= \left(r\left(\frac{-a_1}{(2+r)(1+r)}\right)\right)'\Big|_{r=r_2}$$

$$= \left(r\left(\frac{a_0}{r(1+r)^2(r+2)}\right)\right)'\Big|_{r=r_2}$$

$$= \left(\frac{1}{(1+r)^2(r+2)}\right)'\Big|_{r=r_2}$$

$$= \left(\frac{1}{(r^2+2r+1)(r+2)}\right)'\Big|_{r=r_2}$$

$$= \left(\frac{1}{r^3+2r^2+2r^2+4r+r+2}\right)'\Big|_{r=r_2}$$

$$= \left(\frac{1}{r^3+4r^2+5r+2}\right)'\Big|_{r=r_2}$$

$$= \frac{-3r^2-8r-5}{(r^3+4r^2+5r+2)^2}\Big|_{r=r_2}$$

$$= \frac{-5}{4}$$

$$y_2 = -\log(x)\, y_1 + 1 + x - \frac{5}{4}x^2 + \ldots$$

E.g. Find 2 independent solns to $x^2y'' + (x-2)xy' + (x^2+2)y = 0$ about $x_0 = 0$.

Soln:

X_0 is a singular point because $(x_0)^2 = (0)^2 = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $x^2y'' + x^2y' - 2xy' + x^2y + 2y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1} - \sum_{n=0}^{\infty} 2a_n (n+r) x^{n+r} +$$

$$\sum_{n=0}^{\infty} a_n x^{n+r+2} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

We want all summations to have x^{n+r}.

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} 2a_n (n+r) x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r} +$$

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

Take $n=0$:

$a_0(r)(r-1) - 2a_0(r) + 2a_0 = 0$

$a_0(r(r-1) - 2r + 2) = 0$

$r^2 - r - 2r + 2 = 0$

$r^2 - 3r + 2 = 0$

$(r-2)(r-1) = 0$

$r_1 = 2, \ r_2 = 1$

Take $n=1$:

$a_1(1+r)(r) - 2a_1(1+r) + a_0(r) + 2a_1 = 0$

$a_1((1+r)(r) - 2(1+r) + 2) + a_0(r) = 0$

$a_1(r + r^2 - 2 - 2r + 2) = -a_0(r)$

$a_1(r^2 - r) = -a_0(r)$

$$a_1 = \frac{-a_0(r)}{(r)(r-1)}$$

$$= \frac{-1}{r-1}$$

Take $n \geq 2$:

$a_n(n+r)(n+r-1) - 2a_n(n+r) + a_{n-1}(n+r-1) + a_{n-2} + 2a_n = 0$

$a_n((n+r)(n+r-1) - 2(n+r) + 2) = -a_{n-2} - a_{n-1}(n+r-1)$

Let $a = n+r$

$a(a-1) - 2a + 2$

$a^2 - a - 2a + 2$

$a^2 - 3a + 2$

$(a-2)(a-1)$

$(n+r-2)(n+r-1)$

$$a_n = \frac{-a_{n-2} - a_{n-1}(n+r-1)}{(n+r-2)(n+r-1)}$$

Take $r=2$:

$n=1$:

$$a_1 = \frac{-1}{2-1}$$

$$= -1$$

$n=2$:

$$a_2 = \frac{-a_0 - a_1(1+2)}{(2)(3)}$$

$$= \frac{-1+3}{6}$$

$$= \frac{1}{3}$$

$n=3$:

$$a_3 = \frac{-a_1 - a_2(4)}{(3)(4)}$$

$$= \frac{1 - \frac{4}{3}}{12}$$

$$= \frac{-\frac{1}{3}}{12}$$

$$= \frac{-1}{36}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots \leftarrow r=2$$

$$= x^2 - x^3 + \frac{x^4}{3} - \frac{x^5}{36} + \dots$$

$$y_2 = a \log(x) y_1 + x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r) x^n\right)\Big|_{r=r_2}$$

$$a = \lim_{r \to r_2} (r-r_2) a_N(r)$$

$$= \lim_{r \to 1} (r-1) a_1(r)$$

$$= \lim_{r \to 1} (r-1)\left(\frac{-1}{r-1}\right)$$

$$= \lim_{r \to 1} -1$$

$$= -1$$

$$C_n = \left[(r-r_2) a_n(r)\right]'\Big|_{r=r_2}$$

$$C_1 = \left[(r-1) a_1(r)\right]'\Big|_{r=r_2}$$

$$= \left((r-1)\left(\frac{-1}{r-1}\right)\right)'\Big|_{r=r_2}$$

$$= 0$$

$$C_2 = \left[(r-1) a_2(r)\right]'\Big|_{r=1}$$

$$= \left((r-1) \left(\frac{-a_0 - a_1 (r+1)}{(r)(r+1)} \right) \right)' \Big|_{r=r_2}$$

$$= \left((r-1) \left(\frac{-1 - \left(\frac{-1}{r-1}\right)(r+1)}{(r)(r+1)} \right) \right)' \Big|_{r=r_2}$$

$$= \left(\cancel{(r-1)} \left(\frac{-(r-1) + (r+1)}{(r)(r+1)\cancel{(r-1)}} \right) \right)' \Big|_{r=r_2}$$

$$= \left(\frac{-r+1+r+1}{(r)(r+1)} \right)' \Big|_{r=r_2}$$

$$= \left(\frac{2}{r^2+r} \right)' \Big|_{r=r_2}$$

$$= \frac{-2(2r+1)}{(r^2+r)^2} \Big|_{r=1}$$

$$= \frac{-2(3)}{(2)^2}$$

$$= -\frac{3}{2}$$

$$y_2 = -\log(x) y_1 + x - x^2 - \frac{3}{2} x^3 + \ldots$$

E.g. Find 2 independent solns to $x^2y'' - (x+3)xy' + (x+3)y = 0$ about $x_0 = 0$ and define the domain of the solns.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $x^2y'' - x^2y' - 3xy' + xy + 3y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r+1} - \sum_{n=0}^{\infty} 3a_n (n+r) x^{n+r}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+r+1} + \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0$$

We want all summations to have x^{n+r}

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r} - \sum_{n=0}^{\infty} 3a_n (n+r) x^{n+r}$$

$$+ \sum_{n=1}^{\infty} a_{n-1} x^{n+r} + \sum_{n=0}^{\infty} 3a_n x^{n+r} = 0$$

Take $n=0$:

$a_0(r)(r-1) - 3a_0(r) + 3a_0 = 0$

$a_0(r(r-1) - 3r + 3) = 0$

$a_0(r^2 - r - 3r + 3) = 0$

$a_0(r^2 - 4r + 3) = 0$

$(r-3)(r-1) = 0$

$r_1 = 3, \quad r_2 = 1$

Take $n \geq 1$:

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 3a_n(n+r) + a_{n-1} + 3a_n$$

$$a_n\left((n+r)(n+r-1) - 3(n+r) + 3\right) = a_{n-1}(n+r-1) - a_{n-1}$$

$$= a_{n-1}(n+r-1-1)$$

$$= a_{n-1}(n+r-2)$$

Let $a = n+r$

$a(a-1) - 3a + 3$

$a^2 - a - 3a + 3$

$a^2 - 4a + 3$

$(a-3)(a-1)$

$(n+r-3)(n+r-1)$

$$a_n = \frac{a_{n-1}(n+r-2)}{(n+r-3)(n+r-1)}$$

For $r=3$:

$$a_n = \frac{a_{n-1}(n+1)}{(n)(n+2)}$$

To find the convergence domain, do

$$\lim_{n \to \infty} \frac{a_n}{a_{n-1}}$$

$$= \lim_{n \to \infty} \frac{(n+1)}{n(n+2)}$$

$$= 0$$

$P = 0$, $r = \infty$ **Recall:** $r = \frac{1}{P}$

$\therefore$ The convergence domain is $-\infty < x < \infty$

$n=1$:	$n=2$:	$n=3$:
$a_1 = \frac{a_0(2)}{(1)(3)}$	$a_2 = \frac{a_1(3)}{(8)}$	$a_3 = \frac{a_2(4)}{15}$
$= \frac{2}{3}$	$= \frac{2}{3} \cdot \frac{3}{8}$	$= \frac{1}{4} \cdot \frac{4}{15}$
	$= \frac{1}{4}$	$= \frac{1}{15}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^3 + a_1 x^4 + a_2 x^5 + a_3 x^6 + \dots$$
$$= x^3 + \frac{2x^4}{3} + \frac{x^5}{4} + \frac{x^6}{15} + \dots$$

$$y_2 = a \log(x) y_1 + x^{r_2} \left(1 + \sum_{n=1}^{\infty} c_n(r) x^n \right) \Big|_{r=r_2}$$

$$a = \lim_{r \to r_2} (r-r_2) a_N(r)$$

$$= \lim_{r \to 1} (r-1) a_2(r)$$

$$= \lim_{r \to 1} \cancel{(r-1)} \left(\frac{a_1(r)}{\cancel{(r-1)}(r+1)} \right)$$

$$= \lim_{r \to 1} \left(\frac{a_0(r-1)}{(r-2)\cancel{(r)}} \right) \left(\frac{\cancel{(r)}}{(r+1)} \right)$$

$$= \lim_{r \to 1} \frac{(r-1)}{(r-2)(r+1)}$$

$$= 0$$

$$c_n = \left[(r-r_2) a_n(r) \right]' \Big|_{r=r_2}$$
$$= \left[(r-1) a_n(r) \right]' \Big|_{r=1}$$

$$c_1 = \left((r-1) a_1(r) \right)' \Big|_{r=1}$$
$$= \left((r-1) \left(\frac{a_0(r-1)}{(r-2)(r)} \right) \right)' \Big|_{r=1}$$

$$= \left(\frac{(r-1)^2}{(r-2)(r)} \right)' \Big|_{r=r_1}$$

$$= \left(\frac{(r-1)^2}{r^2-2r} \right)' \Big|_{r=r_1}$$

$$= \frac{2(r-1)(r^2-2r) - (r-1)^2(2r-2)}{(r^2-2r)^2} \Big|_{r=1}$$

$$= 0$$

$$c_2 = [(r-1)a_2(r)]' \big|_{r=1}$$

$$= \left((\cancel{r-1}) \left(\frac{a_1(r)}{(\cancel{r-1})(r+1)} \right) \right)' \Big|_{r=1}$$

$$= \left(\left(\frac{r-1}{(r-2)(\cancel{r})} \right) \left(\frac{\cancel{r}}{r+1} \right) \right)' \Big|_{r=1}$$

$$= \left(\frac{r-1}{r^2-r-2} \right)' \Big|_{r=1}$$

$$= \frac{(r^2-r-2) - (r-1)(2r-1)}{(r^2-r-2)^2} \Big|_{r=1}$$

$$= \frac{-2}{4}$$

$$= -\frac{1}{2}$$

$$y_2 = x - \frac{x^3}{2} + \ldots$$

E.g. Find 2 independent solns to $x^2y'' + xy' + x^2y = 0$ about $x_0 = 0$ and find the convergence domain.

Soln:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $x^2y'' + xy' + x^2y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

Take $n=0$:

$a_0(r)(r-1) + a_0(r) = 0$

$a_0(r^2 - r + r) = 0$

$r^2 = 0$

$r_1 = r_2 = 0$

Take $n \geq 2$:

$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} = 0$

$a_n((n+r)(n+r-1) + (n+r)) = -a_{n-2}$

Let $a = n+r$

$a(a-1) + a$

a^2

$(n+r)^2$

$$a_n = \frac{-a_{n-2}}{(n+r)^2}$$

Since $r=0$, $a_n = \dfrac{-a_{n-2}}{n^2}$

$$\frac{a_n}{a_{n-2}} = -\frac{1}{n^2}$$

$$\lim_{n \to \infty} \frac{a_n}{a_{n-2}}$$

$$= \lim_{n \to \infty} \frac{-1}{n^2}$$

$$= 0$$

$P=0 \rightarrow R=\infty$

∴ The convergence domain is $-\infty < x < \infty$.

Take $n=2$:

$$a_2 = \frac{-a_0}{2^2}$$

$$= -\frac{1}{4}$$

Take $n=4$:

$$a_4 = \frac{-a_2}{4^2}$$

$$= \frac{1}{64}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 + a_1 x^2 + a_2 x^4 + \ldots \leftarrow r=0$$

$$= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \ldots$$

To find y_2, we need to use the Frobenius Method.

$$y_2 = \log(x)\, y_1 + \sum_{n=0}^{\infty} a_n'(r)\, x^{n+r} \Big|_{r=r_1}$$

$$a_2' = \partial r\,(a_2)\,|_{r=r_1}$$
$$= \partial r\left(\frac{-a_0}{(2+r)^2}\right)\Big|_{r=0}$$
$$= \partial r\left(\frac{-1}{(2+r)^2}\right)\Big|_{r=0}$$
$$= \frac{2(2+r)}{(r+2)^4}\Big|_{r=0}$$
$$= \frac{2}{(r+2)^3}\Big|_{r=0}$$
$$= \frac{2}{8}$$
$$= \frac{1}{4}$$

$$a_2' x^2 = \frac{x^2}{4}$$

$$a_4' = \partial r\,(a_4)\,|_{r=r_1}$$
$$= \partial r\left(\frac{-a_2}{(4+r)^2}\right)\Big|_{r=0}$$
$$= \partial r\left(\frac{1}{(2+r)^2(4+r)^2}\right)\Big|_{r=0}$$
$$= \partial r\left(\frac{1}{(r^2+6r+8)^2}\right)\Big|_{r=0}$$
$$= \frac{-2(r^2+6r+8)(2r+6)}{(r^2+6r+8)^4}\Big|_{r=0}$$
$$= \frac{-2(2r+6)}{(r^2+6r+8)^3}\Big|_{r=0}$$
$$= \frac{-2(6)}{8^3}$$
$$= \frac{-12}{512}$$

$$a_4' x^4 = \frac{-12x^4}{512} \qquad y_2 = \log(x) y_1 + \frac{x^2}{4} - \frac{12x^4}{512} + \dots$$

Bessel Eqn

- Has the formula $x^2y'' + xy' + (x^2-v^2)y=0$ where v is a constant called the order of the Bessel eqn or index.

- Bessel eqn of order Zero:

This occurs when $v=0$.
The eqn is now $x^2y'' + xy' + x^2y=0$.

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}, \quad y'=\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1},$$

$$y''=\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}$$

We can rewrite $x^2y'' + xy' + x^2y=0$ as

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} +$$

$$\sum_{n=0}^{\infty} a_n x^{n+r+2} = 0$$

We want all summations to have x^{n+r}.

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} +$$

$$\sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0$$

Take $n=0$:

$a_0(r)(r-1)+a_0(r)=0$

$a_0(r^2-r+r)=0$

$r^2=0$

$r_1=r_2=0$

For $r=0$, take $n=1$:

$a_1(1)(1-1)+a_1(1)=0$

$a_1=0$

For $r=0$, take $n\geq 2$:

$a_n(n+r)(n+r-1)+a_n(n+r)+a_{n-2}=0$

$a_n((n+r)(n+r-1)+(n+r)) = -a_{n-2}$

Let $a=n+r$

$a(a-1)+a$

a^2-a+a

a^2

$(n+r)^2$

$$a_n = \frac{-a_{n-2}}{(n+r)^2}$$

$$= \frac{-a_{n-2}}{n^2} \quad \text{Since } r=0$$

Note: Since $a_1=0$, $a_3, a_5, \ldots, a_{2k+1}=0$.

$n=2$:

$$a_2 = \frac{-a_0}{2^2} = \frac{-1}{4}$$

$n=4$:

$$a_4 = \frac{-a_2}{4^2} = \frac{1}{2^2\cdot 4^2}$$

$n=6$:

$$a_6 = \frac{-a_4}{6^2} = \frac{-1}{2^2\cdot 4^2\cdot 6^2}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + \dots \quad r = 0$$
$$= 1 - \frac{x^2}{4} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

y_1 is called the Bessel function of the first kind of order zero and the standard notation is $J_0(x)$.

To find Y_2, we need to use the Frobenius Method. We know that $r_1 = r_2$.

$$y_2 = \log(x)\, y_1 + \sum_{n=0}^{\infty} a_n'(r)\, x^{n+r} \Big|_{r=r_1}$$

$$a_2' = \partial_r (a_2)$$
$$= \partial_r \left(\frac{-a_0}{(2+r)^2} \right)$$
$$= \partial_r \left(\frac{-1}{(r+2)^2} \right)$$
$$= \frac{2(r+2)}{(r+2)^4}$$
$$= \frac{2}{(r+2)^3}$$

$$a_2' x^{2+r} \Big|_{r=0} = \frac{x^2}{4}$$

$$a_4' = \partial_r (a_4)$$
$$= \partial_r \left(\frac{-a_2}{(4+r)^2} \right)$$
$$= \left(\frac{1}{(2+r)^2 (4+r)^2} \right)'$$
$$= \left(\frac{1}{(r^2+6r+8)^2} \right)'$$
$$= \frac{-2(r^2+6r+8)(2r+6)}{(r^2+6r+8)^4}$$
$$= \frac{-2(2r+6)}{(r^2+6r+8)^3}$$

$$a_4' x^{4+r} \Big|_{r=0} = \frac{-12x^4}{512}$$

$$y_2 = \log(x)\, y_1 + \frac{x^2}{4} - \frac{12}{512} x^4 + \dots$$

y_2 is called the Bessel function of the second kind of order zero and the standard notation is $Y_0(x)$.

The general soln is $y = C_1 J_0(x) + C_2 Y_0(x)$.

- Bessel eqn of order One-Half:

This occurs when $v = \frac{1}{2}$.
The eqn is now $x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 0$.
We can rewrite $x^2y'' + xy' + x^2y - \frac{1}{4}y = 0$ as

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} +$$

$$\sum_{n=2}^{\infty} a_{n-2}x^{n+r} - \sum_{n=0}^{\infty} \frac{1}{4} a_n x^{n+r} = 0$$

Take $n=0$:
$a_0(r)(r-1) + a_0(r) - \frac{1}{4}a_0 = 0$
$a_0(r^2 - r + r - \frac{1}{4}) = 0$
$r^2 = \frac{1}{4}$
$r_1 = \frac{1}{2},\ r_2 = -\frac{1}{2}$

For $r = \frac{1}{2}$, take $n=1$:
$a_1(\frac{3}{2})(\frac{1}{2}) + a_1(\frac{3}{2}) - \frac{1}{4}a_1 = 0$
$\frac{3a_1}{4} + \frac{3a_1}{2} - \frac{a_1}{4} = 0$

$2a_1 = 0$
$a_1 = 0$

For $r = \frac{1}{2}$, take $n \geq 2$:

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \tfrac{1}{4}a_n = 0$$

$$a_n\left((n+r)(n+r-1) + (n+r) - \tfrac{1}{4}\right) = -a_{n-2}$$

Let $a = n+r$

$$a(a-1) + a - \tfrac{1}{4}$$

$$a^2 - a + a - \tfrac{1}{4}$$

$$a^2 - \tfrac{1}{4}$$

$$(a - \tfrac{1}{2})(a + \tfrac{1}{2})$$

$$(n+r-\tfrac{1}{2})(n+r+\tfrac{1}{2})$$

$$a_n = \frac{-a_{n-2}}{(n+r-\frac{1}{2})(n+r+\frac{1}{2})}$$

$$= \frac{-a_{n-2}}{(n)(n+1)}$$

Note: Since $a_1 = 0$, $a_3, a_5, \ldots, a_{2k+1} = 0$

$$a_2 = \frac{-a_0}{(2)(3)} = \frac{-1}{6}$$

$$a_4 = \frac{-a_2}{(4)(5)} = \frac{1}{120}$$

$$a_6 = \frac{-a_4}{(6)(7)} = \frac{-1}{5040}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{1/2} + a_2 x^{5/2} + a_4 x^{9/2} + a_6 x^{13/2} + \ldots$$

$$= x^{1/2} - \frac{x^{5/2}}{6} + \frac{x^{9/2}}{120} - \frac{x^{13/2}}{5040} + \ldots$$

y_1 is called the Bessel function of the first kind of order one-half. The standard notation is $J_{\frac{1}{2}}(x)$.

To find Y_2, we need to use the Frobenius Method.

$$r_1 - r_2 = \tfrac{1}{2} - (-\tfrac{1}{2})$$
$$= 1$$

$N = 1$

$$y_2 = a\log(x)\, y_1 + x^{r_2}\left(1 + \sum_{n=1}^{\infty} C_n(r)\, x^n\right)$$

$$a = \lim_{r \to r_2} (r - r_2)\, a_N(r)$$
$$= \lim_{r \to -\frac{1}{2}} (r + \tfrac{1}{2})\, a_1(r)$$

$$a_1(1+r)(r) + a_1(1+r) - \tfrac{1}{4} a_1 = 0$$
$$a_1\left((1+r)r + 1 + r - \tfrac{1}{4}\right) = 0$$
$$a_1\left(r^2 + 2r + 3/4\right) = 0$$

Since we have r approaching $-\frac{1}{2}$, $r^2 + 2r + 3/4 \neq 0$.
Hence, this means that $a_1 = 0$.
Hence, $a = 0$.

$$C_n = \left[(r - r_2)\, a_n(r)\right]' \Big|_{r = r_2}$$
$$= \left[(r + \tfrac{1}{2})\, a_n(r)\right]' \Big|_{r = -\frac{1}{2}}$$

$$C_2 = \left[(r + \tfrac{1}{2})\, a_2(r)\right]' \Big|_{r = -\frac{1}{2}}$$
$$= \left((r + \tfrac{1}{2})\left(\frac{-a_0}{(r + \frac{3}{2})(r + \frac{5}{2})}\right)\right)' \Bigg|_{r = -\frac{1}{2}}$$
$$= \left(\frac{-r - \frac{1}{2}}{r^2 + 4r + 15/4}\right)' \Bigg|_{r = -\frac{1}{2}}$$
$$= \frac{-r^2 - 4r - 15/4 - (-r - \frac{1}{2})(2r + 4)}{(r^2 + 4r + 15/4)^2} \Bigg|_{r = -\frac{1}{2}}$$

$$= \frac{-(-\frac{1}{2})^2 - 4(-\frac{1}{2}) - \frac{15}{4}}{\left((-\frac{1}{2})^2 + 4(-\frac{1}{2}) + \frac{15}{4}\right)^2}$$

$$= \frac{-\frac{1}{4} + 2 - \frac{15}{4}}{(\frac{1}{4} - 2 + 15/4)^2}$$

$$= \frac{2-4}{(4-2)^2}$$

$$= \frac{-2}{4}$$

$$= \frac{-1}{2}$$

$$C_4 = \left[(r+\tfrac{1}{2})a_4(r)\right]' \Big|_{r=-\frac{1}{2}}$$

$$= \left((r+\tfrac{1}{2})' a_4(r) + (r+\tfrac{1}{2}) a_4'(r)\right) \Big|_{r=-\frac{1}{2}}$$

$$= a_4(-1/2)$$

$$= \frac{-a_2(-1/2)}{(4-1)(4)}$$

$$= \frac{1}{(12)(2)}$$

$$= \frac{1}{24}$$

$$y_2 = x^{-1/2} - \frac{x^{3/2}}{2} + \frac{x^{5/2}}{24} - \ldots$$

y_2 is called the Bessel function of the second kind of order one-half. The standard notation is $Y_{\frac{1}{2}}(x)$.

E.g. Find 2 linearly independent solns of the Bessel eqn of order $\frac{11}{2}$.

Soln:

The Bessel eqn is $x^2y'' + xy' + (x^2 - \frac{121}{4})y = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $x^2y'' + xy' + x^2y - \frac{121}{4}y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

$$- \sum_{n=0}^{\infty} \frac{121}{4} a_n x^{n+r} = 0$$

Take $n=0$:

$a_0 (r)(r-1) + a_0 (r) - \frac{121}{4} a_0 = 0$

$a_0 (r^2 - r + r - \frac{121}{4}) = 0$

$r^2 - \frac{121}{4} = 0$

$r_1 = \frac{11}{2}, \; r_2 = -\frac{11}{2}$

For $r = \frac{11}{2}$, take $n=1$:

$a_1 (1 + \frac{11}{2})(\frac{11}{2}) + a_1 (1 + \frac{11}{2}) - a_1 (\frac{121}{4}) = 0$

$a_1 ((1 + \frac{11}{2})(\frac{11}{2}) + (1 + \frac{11}{2}) - \frac{121}{4}) = 0$

$a_1 = 0$

Take $n \geq 2$:

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - a_n\left(\frac{121}{4}\right) = 0$$

$$a_n\left((n+r)(n+r-1) + (n+r) - \frac{121}{4}\right) = -a_{n-2}$$

Let $a = n+r$

$$a(a-1) + a - \frac{121}{4}$$

$$a^2 - \frac{121}{4}$$

$$\left(a - \frac{11}{2}\right)\left(a + \frac{11}{2}\right)$$

$$\left(n+r - \frac{11}{2}\right)\left(n+r + \frac{11}{2}\right)$$

$$a_n = \frac{-a_{n-2}}{\left(n+r-\frac{11}{2}\right)\left(n+r+\frac{11}{2}\right)}$$

For $r = \frac{11}{2}$, $a_n = \dfrac{-a_{n-2}}{(n)(n+11)}$

Since $a_1 = 0$, $a_{2k+1} = 0$.

$n=2$:

$$a_2 = \frac{-a_0}{(2)(13)} = \frac{-1}{26}$$

$n=4$:

$$a_4 = \frac{-a_2}{(4)(15)} = \frac{1}{26 \cdot 50}$$

$n=6$:

$$a_6 = \frac{-a_4}{(6)(17)} = \frac{-1}{6 \cdot 17 \cdot 26 \cdot 50}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{\frac{11}{2}} + a_2 x^{\frac{15}{2}} + a_4 x^{\frac{19}{2}} + \ldots$$

$$= x^{11/2} - \frac{x^{15/2}}{26} + \frac{x^{19/2}}{26 \cdot 50} - \ldots$$

$$y_2 = a\log(x)\, y_1 + x^{r_2}\left(1+\sum_{n=1}^{\infty} c_n(r)\, x^n\right)\Big|_{r=r_2}$$

$$a = \lim_{r\to r_2} (r-r_2)\, a_N(r), \quad N = r_1 - r_2$$

$$= \tfrac{11}{2} - \left(-\tfrac{11}{2}\right)$$

$$= 11$$

$$= \lim_{r\to -\frac{11}{2}} \left(r+\tfrac{11}{2}\right) a_{11}(r)$$

$$= 0$$

$$c_n = \left[(r-r_2)\, a_n(r)\right]'\Big|_{r=r_2}$$

$$c_2 = \left[\left(r+\tfrac{11}{2}\right) a_2(r)\right]'\Big|_{r=-\frac{11}{2}}$$

$$= \left(\left(r+\tfrac{11}{2}\right)\left(\frac{-a_0}{\left(r-\frac{7}{2}\right)\left(r+\frac{15}{2}\right)}\right)\right)'\Bigg|_{r=-\frac{11}{2}}$$

$$= \left(\frac{-r-\frac{11}{2}}{r^2+4r-\frac{105}{4}}\right)'\Bigg|_{r=-\frac{11}{2}}$$

$$= \frac{(-1)\left(r^2+4r-\frac{105}{4}\right) - \left(-r-\frac{11}{2}\right)(2r+4)}{\left(r^2+4r-\frac{105}{4}\right)^2}\Bigg|_{r=-\frac{11}{2}}$$

$$= \frac{(-1)\left(\frac{121}{4} + 4\left(-\frac{11}{2}\right) - \frac{105}{4}\right) - \left(+\frac{11}{2}-\frac{11}{2}\right)\left(2\left(-\frac{11}{2}\right)+4\right)}{\left(\left(-\frac{11}{2}\right)^2 + 4\left(-\frac{11}{2}\right) - \frac{105}{4}\right)^2}$$

$$= \frac{(-1)(4-22)}{(4-22)^2}$$

$$= \frac{18}{18^2}$$

$$= \frac{1}{18}$$

$$c_4 = [(r-r_2)a_4(r)]'|_{r=r_2}$$
$$= a_4(-\tfrac{11}{2})$$
$$= \frac{-a_2(-\frac{11}{2})}{(4-11)(4)}$$
$$= \frac{1}{(-7)(4)(-9)(2)}$$
$$= \frac{1}{504}$$

$$y_2 = x^{-11/2} + \frac{x^{-7/2}}{18} + \frac{x^{-3/2}}{504} + \dots$$

E.g. Find 2 linearly independent solns of the Bessel eqn of order 1.

Soln:

The Bessel eqn is $x^2y'' + xy' + (x^2-1)y = 0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

We can rewrite $x^2y'' + xy' + x^2y - y = 0$ as

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} a_n (n+r) x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Take $n=0$

$a_0(r)(r-1)+a_0(r)-a_0=0$

$a_0(r^2-r+r-1)=0$

$r^2=1$

$r_1=1,\ r_2=-1$

For $r=1$, take $n=1$

$a_1(2)(1)+a_1(2)-a_1=0$

$3a_1=0$

$a_1=0$

Take $n\geq 2$

$a_n(n+r)(n+r-1)+a_n(n+r)+a_{n-2}-a_n=0$

$a_n((n+r)(n+r-1)+(n+r)-1) = -a_{n-2}$

Let $a=n+r$

$a(a-1)+a-1$

a^2-1

$(a-1)(a+1)$

$(n+r-1)(n+r+1)$

$$a_n = \frac{-a_{n-2}}{(n+r-1)(n+r+1)}$$

For $r=1$, $a_n = \dfrac{-a_{n-2}}{n(n+2)}$

Since $a_1=0$, $a_{2k+1}=0$.

$n=2$:

$$a_2 = \frac{-a_0}{2(4)} = \frac{-1}{8}$$

$n=4$:

$$a_4 = \frac{-a_2}{4\cdot 6} = \frac{1}{192}$$

$n=6$:

$$a_6 = \frac{-a_4}{6\cdot 8} = \frac{-1}{6\cdot 8\cdot 192}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$\begin{aligned} y_1 &= a_0 x + a_2 x^3 + a_4 x^5 + a_6 x^7 + \dots \\ &= x - \frac{x^3}{8} + \frac{x^5}{192} - \frac{x^7}{6 \cdot 8 \cdot 192} + \dots \end{aligned}$$

$$y_2 = a \log(x)\, y_1 + x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r) x^n\right)\Big|_{r=r_2}$$

$$\begin{aligned} a &= \lim_{r \to r_2} (r - r_2)\, a_N(r), \quad N = r_1 - r_2 = 1 - (-1) = 2 \\ &= \lim_{r \to -1} (r+1)\, a_2(r) \\ &= \lim_{r \to -1} (r+1) \left(\frac{-a_0}{(r+1)(r+3)} \right) \\ &= \lim_{r \to -1} \frac{-1}{r+3} \\ &= \frac{-1}{2} \end{aligned}$$

$$c_n = \left[(r - r_2)\, a_n(r)\right]' \Big|_{r=r_2}$$

$$\begin{aligned} c_2 &= \left[(r+1)\, a_2(r)\right]' \Big|_{r=-1} \\ &= \left((r+1) \left(\frac{-a_0}{(r+1)(r+3)} \right) \right)' \Big|_{r=-1} \\ &= \left(\frac{-1}{r+3} \right)' \Big|_{r=-1} \\ &= \frac{1}{(r+3)^2} \Big|_{r=-1} \\ &= \frac{1}{4} \end{aligned}$$

$$c_4 = [(r+1)a_4(r)]'\Big|_{r=r_2}$$

$$= \left((r+1)\left(\frac{-a_2}{(4-2)4}\right)\right)'\Big|_{r=-1}$$

$$= \left((r+1)\left(\frac{1}{8(r+1)(r+3)}\right)\right)'\Big|_{r=-1}$$

$$= \left(\frac{1}{8r+24}\right)'\Big|_{r=-1}$$

$$= \frac{-8}{(8r+24)^2}\Big|_{r=-1}$$

$$= \frac{-8}{16^2}$$

$$= \frac{-1}{32}$$

$$y_2 = -\frac{1}{2}\log(x)y_1 + x^{-1} + \frac{x}{4} - \frac{x^3}{32} + \dots$$

E.g. Find 2 linearly independent solns of the Bessel eqn of order $\frac{3}{2}$.

Soln:

Bessel eqn: $x^2y'' + xy' + (x^2 - \frac{9}{4})y = 0$

We can rewrite it as

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r}$$

$$+ \sum_{n=0}^{\infty} -\frac{9}{4}a_n x^{n+r} = 0$$

Take $n=0$:

$a_0(r)(r-1)+a_0(r)-a_0(\frac{9}{4})=0$

$a_0(r^2-r+r-\frac{9}{4})=0$

$r^2=\frac{9}{4}$

$r_1=\frac{3}{2},\quad r_2=-\frac{3}{2}$

For $r=\frac{3}{2}$, take $n=1$

$a_1(\frac{5}{2})(\frac{3}{2})+a_1(\frac{3}{2})-a_1(\frac{9}{4})=0$

$a_1(\frac{15}{4}+\frac{3}{2}-\frac{9}{4})=0$

$a_1=0$

Take $n\geq 2$:

$a_n(n+r)(n+r-1)+a_n(n+r)+a_{n-2}-\frac{9}{4}a_n=0$

$a_n((n+r)(n+r-1)+(n+r)-\frac{9}{4})=-a_{n-2}$

Let $a=n+r$

$a(a-1)+a-\frac{9}{4}$

$a^2-\frac{9}{4}$

$(a-\frac{3}{2})(a+\frac{3}{2})$

$(n+r-\frac{3}{2})(n+r+\frac{3}{2})$

$$a_n=\frac{-a_{n-2}}{(n+r-\frac{3}{2})(n+r+\frac{3}{2})}$$

For $r=\frac{3}{2}$, $a_n=\dfrac{-a_{n-2}}{(n)(n+3)}$

Since $a_1=0$, $a_{2k+1}=0$

$n=2$:

$a_2=\dfrac{-a_0}{(2)(5)}$

$=\dfrac{-1}{10}$

$n=4$:

$a_4=\dfrac{-a_2}{(4)(7)}$

$=\dfrac{1}{280}$

$n=6$:

$a_6=\dfrac{-a_4}{(6)(9)}$

$=\dfrac{-1}{54\cdot 280}$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1 = a_0 x^{3/2} + a_2 x^{7/2} + a_4 x^{11/2} + \ldots$$
$$= x^{3/2} - \frac{x^{7/2}}{10} + \frac{x^{11/2}}{280} - \ldots$$

$$r_1 - r_2 = \tfrac{3}{2} - (-\tfrac{3}{2})$$
$$= 3$$

$N = 3$

$$y_2 = a \log(x) y_1 + x^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r) x^n\right) \Big|_{r=r_2}$$

$$a = \lim_{r \to r_2} (r - r_2) a_N(r)$$
$$= \lim_{r \to -\frac{3}{2}} (r + \tfrac{3}{2}) a_3(r)$$
$$= 0$$

$$c_n = \left[(r - r_2) a_n(r)\right]' \Big|_{r=r_2}$$

$$c_2 = \left[(r + \tfrac{3}{2}) a_2(r)\right]' \Big|_{r=-3/2}$$
$$= \left((r + \tfrac{3}{2}) \left(\frac{-a_0(r)}{(n+r-3/2)(n+r+\frac{3}{2})} \right) \right)' \Big|_{r=-\frac{3}{2}}$$
$$= \frac{-1}{(2-3)(2)}$$
$$= \frac{1}{2}$$

$$y_2 = x^{-3/2} + \frac{x^{1/2}}{2} + \ldots$$

More Questions

E.g. Solve $x^2y'' - 3xy' + 4y = x^2\log x$ using variation of parameters

Soln:

First, we solve $x^2y'' - 3xy' + 4y = 0$ to solve for y_1 and y_2.

$x^2y'' - 3xy' + 4y = 0$ ← Euler Eqn

$\alpha = -3, \beta = 4$

$r^2 + (\alpha - 1)r + \beta = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r_1 = 2$

$y_1 = x^2, \quad y_2 = \log(x) \cdot x^2$

$U_1'y_1 + U_2'y_2 = 0$

$U_1'y_1' + U_2'y_2' = x^2\log x$

$$U_1' = \frac{-U_2'y_2}{y_1}$$

$$= \frac{-U_2'x^2\log(x)}{x^2}$$

$$= -U_2'\log(x)$$

$(-U_2'\log(x))(2x) + U_2'(2x\log(x) + x) = x^2\log x$

$U_2'(2x)(\log x) + U_2'(x) - U_2'(2x)(\log x) = x^2\log x$

$U_2'(x) = x^2\log x$

$U_2' = x\log x$

$U_2 = \int x\log x \, dx$

Let $u = \log x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = \frac{du}{x}$

$$\begin{aligned} u_2 &= \int u\, du \\ &= \frac{u^2}{2} + C_2 \\ &= \frac{\log^2(x)}{2} + C_2 \end{aligned}$$

$$\begin{aligned} u_1' &= -u_2' \log(x) \\ &= -x \log^2(x) \end{aligned}$$

$u_1 = \int -x \log^2(x)\, dx$

Let $u = \log(x)$

$\frac{du}{dx} = \frac{1}{x}$

$dx = \frac{du}{x}$

$$\begin{aligned} u_1 &= \int -u^2\, du \\ &= \frac{-u^3}{3} + C_1 \\ &= \frac{-\log^3(x)}{3} + C_1 \end{aligned}$$

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= \left(\frac{-\log^3(x)}{3} + C_1\right) x^2 + \left(\frac{\log^2(x)}{2} + C_2\right) \log(x) \cdot x^2 \end{aligned}$$

E.g. Solve $x' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \bar{x} + \begin{bmatrix} 2e^{-3t} \\ 3te^{-3t} \end{bmatrix}$

Soln:

$\begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = 0$

$(1-r)(-7-r)+16=0$

$-7-r+7r+r^2+16=0$

$r^2+6r+9=0$

$(r+3)^2=0$

$r_1=r_2=-3$

$(A-rI)\bar{z}=\bar{0}$

when $r=-3$

$\begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$4z_1 - 4z_2 = 0$

$4z_1 - 4z_2 = 0$ ← Redundant

$4z_1 = 4z_2$

$z_1 = z_2$

let $z_1=1.$ $z_2=1$

$\bar{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

To find the second eigenvector, we need to use a generalized eigenvector.

$(A - rI)\bar{p} = \bar{z}$

When $r = 3$

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$4p_1 - 4p_2 = 1$

$4p_1 - 4p_2 = 1$ ← Redundant

Let $p_2 = 0$

$p_1 = 1/4$

$$\bar{p} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$$

The soln to the H system is $\bar{x} = c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} +$

$$c_2 \left(te^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \right)$$

Now, we want to solve for u_1 and u_2 in

$$u_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + u_2 \left(te^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \right)$$

$$U_1' e^{-3t}\begin{bmatrix}1\\1\end{bmatrix} + U_2'\left(te^{-3t}\begin{bmatrix}1\\1\end{bmatrix} + e^{-3t}\begin{bmatrix}1/4\\0\end{bmatrix}\right) = \begin{bmatrix}2e^{-3t}\\3te^{-3t}\end{bmatrix}$$

$$U_1'e^{-3t} + U_2'(te^{-3t} + \tfrac{1}{4}e^{-3t}) = 2e^{-3t}$$
$$U_1'e^{-3t} + U_2'te^{-3t} = 3te^{-3t}$$

$$U_1' + U_2'(t + \tfrac{1}{4}) = 2$$
$$U_1' + U_2't = 3t$$

$$U_1' + U_2't + \tfrac{1}{4}U_2' = 2 \quad (1)$$
$$U_1' + U_2't = 3t \quad (2)$$

Do (1) − (2)

$$\frac{U_2'}{4} = 2-3t$$

$$U_2' = 8-12t$$
$$U_2 = \int 8-12t \, dt$$
$$= 8t - 6t^2 + C_2$$

$$U_1' + U_2't = 3t$$
$$U_1' = 3t - (8-12t)t$$
$$= 3t - 8t + 12t^2$$
$$= -5t + 12t^2$$
$$U_1 = \int -5t + 12t^2 \, dt$$
$$= \frac{-5t^2}{2} + 4t^3 + C_1$$

E.g. Solve $\bar{x}' = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} \bar{x} + \begin{bmatrix} e^{2t} \cos(t) \\ e^{2t} \sin(t) \end{bmatrix}$

Soln:

$$\begin{vmatrix} 2-r & 3 \\ -3 & 2-r \end{vmatrix} = 0$$

$(2-r)^2 + 9 = 0$

$r^2 - 4r + 4 + 9 = 0$

$r^2 - 4r + 13 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 \pm 3i$$

Take $r = 2 + 3i$

$(A - rI)\bar{z} = \bar{0}$

$$\begin{bmatrix} 2-(2+3i) & 3 \\ -3 & 2-(2+3i) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3i z_1 + 3z_2 = 0$

$-3z_1 - 3i z_2 = 0$

$i z_1 = z_2$

ut $z_1 = 1$. $z_2 = 2$

$$\bar{z}' = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e^{r_1 t}\overline{z_1} = e^{(2+3i)t}\overline{z_1}$$

$$= e^{2t}\cdot \quad e^{(3t)i}\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= e^{2t}\left(\cos(3t) + i\sin(3t)\right)\left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= e^{2t}\left(\cos(3t)\begin{bmatrix}1\\0\end{bmatrix} - \sin(3t)\begin{bmatrix}0\\1\end{bmatrix}\right) +$$

$$ie^{2t}\left(\cos(3t)\begin{bmatrix}0\\1\end{bmatrix} + \sin(3t)\begin{bmatrix}1\\0\end{bmatrix}\right)$$

The gen soln to the H system is

$$\bar{x} = C_1 e^{2t}\left(\cos(3t)\begin{bmatrix}1\\0\end{bmatrix} - \sin(3t)\begin{bmatrix}0\\1\end{bmatrix}\right) +$$

$$C_2 e^{2t}\left(\cos(3t)\begin{bmatrix}0\\1\end{bmatrix} + \sin(3t)\begin{bmatrix}1\\0\end{bmatrix}\right)$$

Now, we'll use variation of parameters to find U_1 and U_2.

$$U_1' e^{2t}\left(\cos(3t)\begin{bmatrix}1\\0\end{bmatrix} - \sin(3t)\begin{bmatrix}0\\1\end{bmatrix}\right) +$$

$$U_2' e^{2t}\left(\cos(3t)\begin{bmatrix}0\\1\end{bmatrix} + \sin(3t)\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}e^{2t}\cos(t)\\e^{2t}\sin(t)\end{bmatrix}$$

$$U_1' e^{2t}(\cos(3t)) + U_2' e^{2t} \sin(3t) = e^{2t} \cos(t)$$
$$-U_1' e^{2t} \sin(3t) + U_2' e^{2t} \cos(3t) = e^{2t} \sin(t)$$

$$U_1' \cos(3t) + U_2' \sin(3t) = \cos(t)$$
$$-U_1' \sin(3t) + U_2' \cos(3t) = \sin(t)$$

$$U_1' = \frac{\cos(t) - U_2' \sin(3t)}{\cos(3t)}$$

$$-\left(\frac{\cos(t) - U_2' \sin(3t)}{\cos(3t)}\right) \sin(3t) + U_2' \cos(3t) = \sin(t)$$

$$(-\cos(t) + U_2' \sin(3t)) \sin(3t) + U_2' \cos^2(3t) = \sin(t)\cos(3t)$$
$$U_2'(\sin^2(3t) + \cos^2(3t)) = \sin(t)\cos(3t) + \cos(t)\sin(3t)$$
$$U_2' = \sin(t)\cos(3t) + \cos(t)\sin(3t)$$
$$= \sin(4t)$$
$$U_2 = \int \sin(4t)\, dt$$
$$= \frac{-\cos(4t)}{4} + C_2$$

$$U_1' = \frac{\cos(t) - \sin(4t)\sin(3t)}{\cos(3t)}$$

$$U_1 = \int \frac{\cos(t) - \sin(4t)\sin(3t)}{\cos(3t)}\, dt$$

E.g. Find the general soln of $\bar{x}' = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \bar{x}$

Soln:

$\begin{vmatrix} 1-r & 2 \\ 4 & -1-r \end{vmatrix} = 0$

$(1-r)(-1-r) - 8 = 0$

$-1 - r + r + r^2 - 8 = 0$

$r^2 - 9 = 0$

$r^2 = 9$

$r = \pm 3$

$r_1 = 3, \quad r_2 = -3$

$(A - rI)\bar{z} = \bar{0}$

when $r = 3$

$\begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2z_1 + 2z_2 = 0$

$4z_1 - 4z_2 = 0$ ← Redundant

$-2z_1 = -2z_2$

$z_1 = z_2$

Let $z_1 = 1. \ z_2 = 1$

$\overline{z_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

when $r=-3$

$$\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4z_1 + 2z_2 = 0$

$4z_1 + 2z_2 = 0$ ← Redundant

$4z_1 = -2z_2$

$-2z_1 = z_2$

let $z_1 = 1.$ $z_2 = -2$

$$\bar{z}^2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The general soln is $\bar{x} = C_1 e^{r_1 t}\,\bar{z}^1 + C_2 e^{r_2 t}\,\bar{z}^2$

$$= C_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

E.g. Find the gen soln of $\bar{x}' = \begin{bmatrix} 4 & -2 & -4 \\ -3 & 5 & 8 \\ 2 & -2 & -3 \end{bmatrix} \bar{x}$

Soln:

$$\begin{vmatrix} 4-r & -2 & -4 \\ -3 & 5-r & 8 \\ 2 & -2 & -3-r \end{vmatrix} = 0$$

$$(4-r)\left[(5-r)(-3-r)+16\right] + 2\left[(-3)(-3-r)-16\right] - 4\left[6-2(5-r)\right] = 0$$

$$(4-r)\left[-15-5r+3r+r^2+16\right] + 2\left[+9+3r-16\right] - 4\left[6-10+2r\right] = 0$$

$$(4-r)\left[r^2-2r+1\right] + 2\left[3r-7\right] - 4\left[2r-4\right] = 0$$

$4r^2 - 8r + 4 - r^3 + 2r^2 - r + 6r - 14 - 8r + 16 = 0$

$-r^3 + 6r^2 - 11r + 6 = 0$

Take $r=1$

$$-(1)^3+6(1)^2-11(1)+6 = -1+6-11+6$$
$$= -12+12$$
$$=0$$

$\therefore$ $r-1$ is a root.

$$\begin{array}{r l} & -r^2+5r-6 \\ r-1 \,\big) & -r^3+6r^2-11r+6 \\ & -(-r^3+r^2) \\ & 5r^2-11r \\ & -(5r^2-5r) \\ & -6r+6 \\ & -(-6r+6) \\ & 0 \end{array}$$

$$(r-1)(-r^2+5r-6) = -r^3+6r^2-11r+6$$
$$=0$$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-5 \pm \sqrt{25-24}}{-2}$$
$$= \frac{-5 \pm 1}{-2}$$
$$= 2, 3$$

$r_1 = 1,\ r_2 = 2,\ r_3 = 3$

$(A - rI)\bar{z} = \bar{0}$

when $r=1$

$$\begin{bmatrix} 3 & -2 & -4 \\ -3 & 4 & 8 \\ 2 & -2 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3z_1 - 2z_2 - 4z_3 = 0$ (1)

$-3z_1 + 4z_2 + 8z_3 = 0$ (2)

$2z_1 - 2z_2 - 4z_3 = 0$ (3)

If you do $\frac{(1) - (2)}{3}$, you get (3)

Hence, (3) is redundant.

$2z_2 = 3z_1 - 4z_3$

$-3z_1 + 2(3z_1 - 4z_3) + 8z_3 = 0$

$-3z_1 + 6z_1 - 8z_3 + 8z_3 = 0$

$3z_1 = 0$

$z_1 = 0$

$2z_2 = -4z_3$

$z_2 = -2z_3$

at $z_3 = 1$. $z_2 = -2$

$$\bar{z^1} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

When $r=2$

$$\begin{bmatrix} 2 & -2 & -4 \\ -3 & 3 & 8 \\ 2 & -2 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2z_1 - 2z_2 - 4z_3 = 0$ (1)
$-3z_1 + 3z_2 + 8z_3 = 0$ (2)
$2z_1 - 2z_2 - 5z_3 = 0$ (3)

If you do (2)((2)+(2)(3)), you get (1).
$2(-3+4)=2$
$2(3-4)=-2$
$2(8-10)=-4$
Hence, (1) is redundant.

$2z_2 = 2z_1 - 5z_3$
$z_2 = z_1 - \frac{5}{2}z_3$

$-3z_1 + 3(z_1 - \frac{5}{2}z_3) + 8z_3 = 0$
$-\frac{15z_3}{2} + 8z_3 = 0$

$z_3 = 0$

$z_2 = z_1$
Let $z_2 = 1$. $z_1 = 1$.

$$\overline{z^2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

When $r=3$

$$\begin{bmatrix} 1 & -2 & -4 \\ -3 & 2 & 8 \\ 2 & -2 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z_1 - 2z_2 - 4z_3 = 0 \quad (1)$$
$$-3z_1 + 2z_2 + 8z_3 = 0 \quad (2)$$
$$2z_1 - 2z_2 - 6z_3 = 0 \quad (3)$$

If you do $\frac{(1) - (2)}{2}$, you get (3).

Hence, (3) is redundant.

$2z_2 = z_1 - 4z_3$

$-3z_1 + z_1 - 4z_3 + 8z_3 = 0$

$-2z_1 + 4z_3 = 0$

$-2z_1 = -4z_3$

$z_1 = 2z_3$

Let $z_3 = 1$. $z_1 = 2$

$2z_2 = 2 - 4$

$= -2$

$z_2 = -1$

$$\overline{z^3} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ The gen soln is $\bar{x} = c_1 e^{t} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$